Physics-Based Stochastic Optimization

Theory and Methods

ICSP 2025 Tutorial

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University of Hamburg · Department of Mathematics

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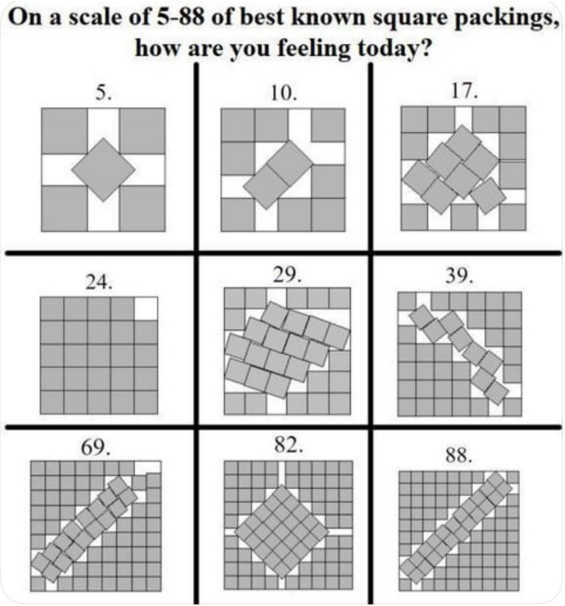
On a scale of 5-88 of best known square packings,
how are you feeling today?
5.
10.
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Personal introduction

2010-2016	BSc., MSc., Technical University of Vienna
2016-2020	PhD, University of Vienna (Advisor: Georg Pflug)
2020-2024	Postdoc at Weierstrass Institute in Berlin (Group: Michael Hintermüller)
2024-2025	Tenure-track assistant professor, University of Hamburg
Fall 2025-	Full professor, Alpen-Adria University of Klagenfurt

Vienna Graduate School for Computational Optimization.

Today's agenda

1. Introduction
   - Challenges and examples of applications in physics-based optimization.
2. Foundations of stochastic optimization on Banach spaces
   - Basic definitions, most useful tools and results for optimization.
3. Optimality conditions
   - Constraints on the first and second stage. Adjoint method.
4. Case study
   - Analysis of example from PDE-constrained optimization under uncertainty.
5. Stochastic approximation
   - Results in Hilbert spaces, handling of numerical error.

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Stochastic optimization

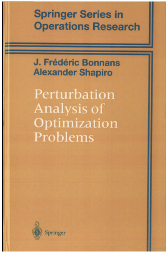
Abstract decision problem

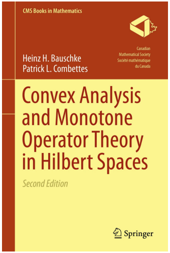
Prototypical **decision problem** from stochastic optimization:

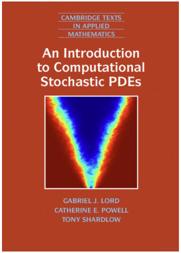
$$\min_{u \in U_{\text{ad}}} \quad \mathbb{E}[J(u,\xi)] = \int_\Omega J(u,\xi(\omega)) \, \mathrm{d}\mathbb{P}(\omega).$$

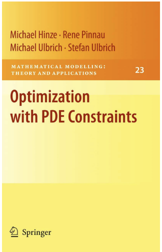
- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be given.
- $U_{\text{ad}} \subset U$ is the *feasible set* of deterministic decisions with U **Banach space.**
- $\xi \colon \Omega \to \Xi \subset \mathbb{R}^m$ is the *random (i.e., measurable) vector* with *support* Ξ.
- $J \colon U \times \Xi \to \mathbb{R}$ is the parametrized objective function; the superposition $\mathcal{J}(u,\omega) := J(u,\xi(\omega))$ is the *random variable objective function*.

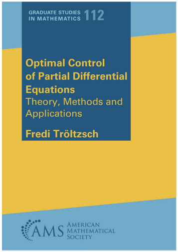
Literature

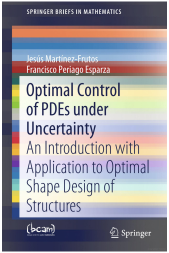

Springer Series in Operations Research
J. Frédéric Bonnans
Alexander Shapiro
Perturbation Analysis of Optimization Problems
Springer


CMS Books in Mathematics
Heinz H. Bauschke
Patrick L. Combettes
Convex Analysis and Monotone Operator Theory in Hilbert Spaces
Second Edition
Springer

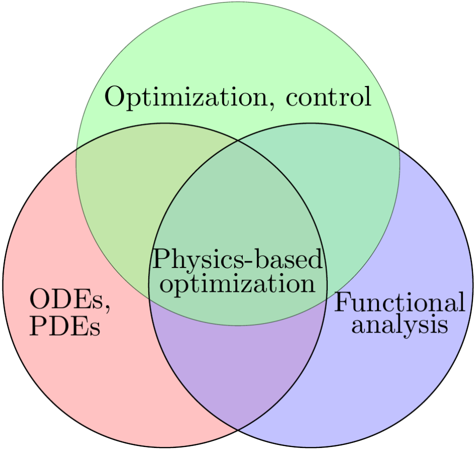

CAMBRIDGE TEXTS IN APPLIED MATHEMATICS
An Introduction to Computational Stochastic PDEs
GABRIEL J. LORD
CATHERINE E. POWELL
TONY SHARDLOW


Michael Hinze · Rene Pinnau
Michael Ulbrich · Stefan Ulbrich
MATHEMATICAL MODELLING: THEORY AND APPLICATIONS
23
Optimization with PDE Constraints
Springer


GRADUATE STUDIES IN MATHEMATICS 112
Optimal Control of Partial Differential Equations
Theory, Methods and Applications
Fredi Tröltzsch
AMS AMERICAN MATHEMATICAL SOCIETY


SPRINGER BRIEFS IN MATHEMATICS
Jesús Martínez-Frutos
Francisco Periago Esparza
Optimal Control of PDEs under Uncertainty
An Introduction with Application to Optimal Shape Design of Structures
bcam
Springer

Physics-based optimization



Example optimal control problem

Boundary control problem for the 1D wave equation

Example

- An oscillating string is fixed at both ends of the interval $(0, \ell)$.
- The state y is the solution to the wave equation:

$$\begin{aligned} \text{state:} \quad & y_{tt}(x,t) - c^2 y_{xx}(x,t) = 0 && \text{in } (0,\ell) \times (0,T), \\ \text{boundary conditions:} \quad & y(0,t) = u_1(t),\ y(\ell,t) = u_2(t) && \text{for } t \in (0,T), \\ \text{initial conditions:} \quad & y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) && \text{for } x \in (0,\ell), \end{aligned}$$

  where c is the wave speed.
- **Goal**: drive the string to rest at a given time T by controls u_1 and u_2 at $x = 0$ and $x = \ell$, respectively, i.e., to minimize the cost functional

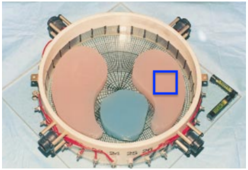
$$\min_{(u,y) \in U \times Y} J(u,y) = \frac{1}{2} \|y\|^2_{L^2(0,T;(0,\ell))} + \frac{1}{2} \|y(\cdot, T)\|^2_{L^2(0,\ell)} + \frac{\mu}{2} \|u\|^2_{L^2(0,T)^2}$$

  where $\mu > 0$ is a regularization parameter.

Simulation: `https://caroline.geiersbach.com/icsp/wave`. (Credit: Felix Sauer, Weierstrass Institute)

Uncertainty in physics-based optimization (1/2)

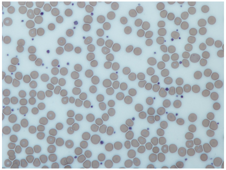
- **Material parameters** (conductivity, permittivity, elasticity)



Left: Experimental setup.

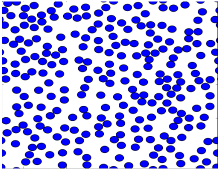
Right: Simulation of admittivity (conductivity + permittivity) of tissue sample using Karhunen-Loève expansion.

Left: Cheney et al. **Electrical impedance tomography** (1999).



Left: CC image showing blood smear with platelets (purple) and red blood cells (gray).

Right: Numerical simulation using randomly generated circles with periodic boundary.



Uncertainty in physics-based optimization (2/2)

- **Boundary conditions** (current density, forcing)

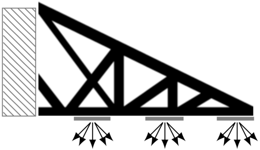
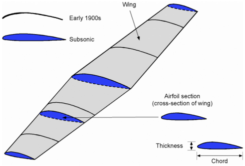


Figure showing cantilever attached to wall on left-hand side and subjected to different scenarios of forces on the lower boundary.

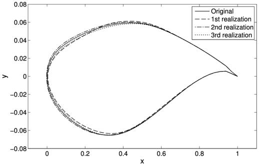
Conti et al. **Stochastic dominance constraints in elastic shape optimization** (2016).

- **Manufacturing irregularites** (random boundary)



Left: Graphic of airfoil.

Right: Simulation with randomly perturbed boundaries.



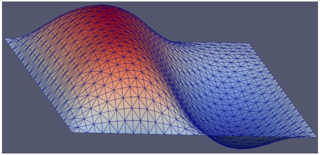
Left: CC; *Right*: Liu et al. **Quantification of airfoil geometry-induced aerodynamic uncertainties-comparison of approaches** (2017).

Application in control of stationary heat source under uncertainty

$$\min_{(u,y)\in U\times\mathcal{Y}} \frac{1}{2}\mathbb{E}\left[\|y-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2$$

$$\text{s.t.} \; -\nabla\cdot(a(x,\omega)\nabla y(x,\omega)) = u(x), \;\text{ in } D \text{ a.s.},$$
$$y(x,\omega) = 0, \qquad \text{in } \partial D \text{ a.s.}$$

$y\ldots$ random state (temperature),
$u\ldots$ deterministic control (optimization variable).



Realization of temperature $y(\cdot,\omega)$ on domain D.

Interpretation as convex stochastic optimization problem:

- Definition of linear control-to-state map $S(u,\omega)=y(\cdot,\omega)\in Y$ using PDE theory.
- Objective $u\mapsto \frac{1}{2}\mathbb{E}\left[\|S(u,\cdot)-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2$ depends only on u.
- U, Y are **Banach spaces**.

a.s. $=$ almost surely with respect to $\mathbb{P}$.

Variants of problem from previous slide

$$\min_{(u,y)\in \boldsymbol{U}_{\text{ad}}\times \boldsymbol{\mathcal{Y}}_{\text{ad}}} \frac{1}{2}\mathcal{R}\left[\|y-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2$$

$$\text{s.t.} -\nabla\cdot(a(x,\omega)\nabla y(x,\omega)) + \boldsymbol{y^3(x)} = u(x), \text{ in } D \text{ a.s.,}$$

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}}(\boldsymbol{x},\omega) = \boldsymbol{g}(\boldsymbol{x},\omega), \quad \text{in } \partial D \text{ a.s.}$$

Examples of popular variants in the literature:

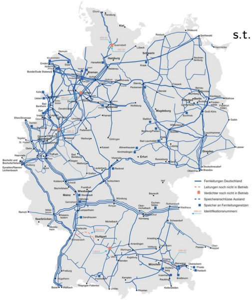
- **Additional constraints** (control or state constraints)
- **Risk measures** (e.g. AV@R)
- **Nonlinearities, other boundary conditions**

Application in gas markets

Subproblem of an N-player noncooperative game:

- Agent i makes decisions $u_i = (p_i^{\text{in}}, p_i^{\text{out}}, q_i^{\text{in}}, q_i^{\text{out}})$ on boundary nodes to minimize loss, while simultaneously satisfying operational constraints.
- System is driven by the collective decisions $u = (u_i)_{i=1}^N$ of all agents.
- Price π depends on decisions of all agents.

$$\max_{u_i \in U_{i,\text{ad}}} \quad \mathbb{E}\left[\int_0^T \pi\left(t, q^{\text{out}}(t), \cdot\right) q_i^{\text{out}}(t) - c(t) q_i^{\text{in}}(t)\, \mathrm{d}t\right] \quad \textbf{[Maximize profit]}$$



$$\left.\begin{aligned} \text{s.t.} \quad & \frac{\partial p_\omega}{\partial t} + \frac{c^2}{A}\frac{\partial q_\omega}{\partial x} = 0, \\ & \frac{\partial q_\omega}{\partial t} + A\frac{\partial p_\omega}{\partial x} = -\frac{\lambda(\omega) c^2}{2dA}\frac{q_\omega |q_\omega|}{p_\omega} - \frac{Ag \sin\alpha}{c^2} p_\omega, \\ & \text{boundary conditions depending on } (u_1, \ldots, u_N), \end{aligned}\right\} \quad \textbf{[PDE describing gas transport]}$$

$$\left.\begin{aligned} \underline{p} \le p_\omega(t,x) \le \overline{p}, \\ \underline{q} \le q_\omega(t,x) \le \overline{q} \quad \text{in } [0,T] \times D \text{ a.s.} \end{aligned}\right\} \quad \textbf{[State constraints]}$$

TRR 154

Theoretical questions (1/2)

And potential difficulties in infinite dimensions

Problem

$$\min_{u \in U_{\text{ad}}} \quad \mathbb{E}[J(u,\xi)] = \int_\Omega J(u,\xi(\omega))\, \mathrm{d}\mathbb{P}(\omega).$$

1. Existence of solutions

Usual argument:

- Let j^* be optimal value (assumed to be attainable). Choose minimizing sequence (u^n) with $\lim_{n\to\infty} \mathbb{E}[J(u^n,\xi)] = j^*$
- Show (u^n) bounded $\Rightarrow$ there exists a convergent subsequence (u^{n_k}) such that $u^{n_k} \to \bar{u}$.
  $\uparrow$ ***only true in finite dimensions.***
- Use continuity to conclude that $\mathbb{E}[J(\bar{u},\xi)] = \lim_{k\to\infty} \mathbb{E}[J(u^{n_k},\xi)] = j^*$; feasibility of $\bar{u}$ using closedness of U_{ad}.

2. Uniqueness of solutions

With strict convexity (easily transferable to infinite dimensions).

Theoretical questions (2/2)

And potential difficulties in infinite dimensions

3. Optimality conditions (constrained problems)

Existence of Lagrange multipliers with (e.g. LI, Slater) constraint qualification.
(CQs generally not transferable to infinite dimensions; derivative needs generalization.)

4. Algorithms

Convergence of methods, e.g. using stochastic gradient:

$$u^{n+1} = u^n - t_n G(u^n, \xi^n), \quad G(u^n, \xi^n) \approx \nabla_u \mathbb{E}[G(u^n, \xi)].$$

(Generally no explicit formula for G; need for discretization.)

Two-norm discrepancy (1/2)

- The two-norm discrepancy is a phenomenon in optimal control in infinite dimensions due to fact that **not all norms are equivalent**. [1]
- Typical situation: **second-order derivative is coercive in weaker norm** than where it is (twice) continuously differentiable ($\rightarrow$ next slide).
- Convergence of optimization algorithms in this setting need to handle this difficulty.

[1] Casas, Tröltzsch. **Second order analysis for optimal control problems: improving results expected from abstract theory** (2012).
Ioffe. **Necessary and sufficient conditions for a local minimum. 3: Second order conditions and augmented duality** (1979).

Two-norm discrepancy (2/2)

Example ([2])

$$\min_{x\in L^2(0,1)} J(x) = \int_0^1 \sin(x(t))\ \mathrm{d}t.$$

Formally, $J''(\bar{x})v^2 = -\int_0^1 \sin(\bar{x}(t))v^2(t)\ \mathrm{d}t = \|v\|^2_{L^2(0,1)}$ at global solution $\bar{x}(t) = -\frac{\pi}{2}$.

Other global solutions:

$$x_\varepsilon(t) = \begin{cases} -\frac{\pi}{2}, & \text{if } t \in [0, 1-\varepsilon], \\ \frac{3\pi}{2}, & \text{if } t \in (1-\varepsilon, 1] \end{cases} \quad \forall \varepsilon \in (0,1).$$

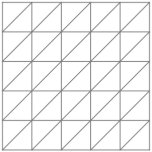
$\Rightarrow$ infinitely many global solutions in L^2-neighborhood of $\bar{x}$.

Problem: J is not C^2 in $L^2(0,1)$ but rather $L^\infty(0,1)$; but $J''(\bar{x})v^2 \geq c\|v\|^2_{L^\infty}$ not valid.

[2] Casas, Tröltzsch. **Second order analysis for optimal control problems: improving results expected from abstract theory** (2012).

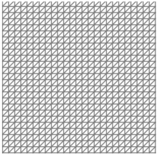
Mesh dependence (1/2)

- In numerical simulations, infinite-dimensional problems without a closed-form solution need to be discretized.
- "Mesh independence" in methods refers to the property that convergence behavior should be invariant w.r.t. increasingly finer discretizations.



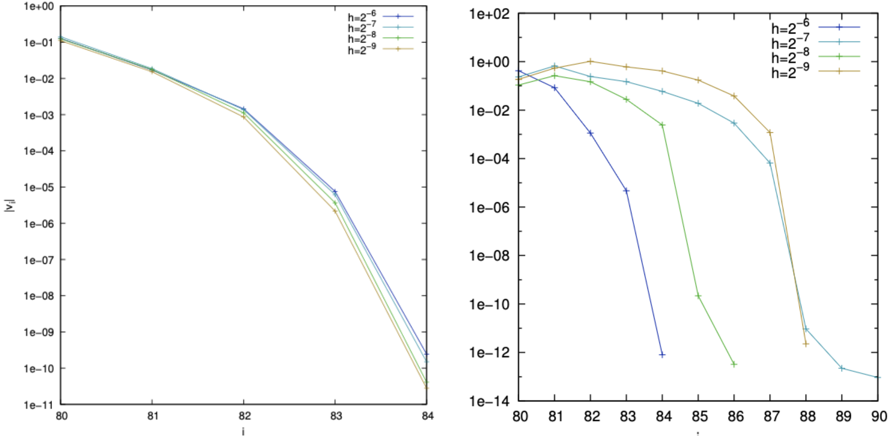
Two different meshes used in finite element method.

⟵ coarse fine ⟶



- Mesh dependence can be observed, for example, if the inner product used for the discretized gradient is inconsistent with the correct inner product in the functional-analytic setting.

Mesh dependence (2/2)



Left: Residual as a function of iteration number with mesh independence; *Right*: with mesh dependence.
From C. Rupprecht. **Projection type methods in Banach space with application in topology optimization**, PhD thesis, 2016.

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Banach space

A (real) **Banach space** $(X, \|\cdot\|_X)$ is a complete normed space.

Example

- $L^p(D) = \{\text{Lebesgue-m.b. } u\colon D \to \mathbb{R} \mid \int_D |u(x)|^p \, \mathrm{d}x < \infty\}/\sim$, $p \in [1, \infty)$, $D \subset \mathbb{R}^d$.
- $L^p_\mu(\Omega) = \{\mathcal{F}\text{-m.b. } u\colon \Omega \to \mathbb{R} \mid \int_\Omega |u(\omega)|^p \, \mathrm{d}\mu(\omega) < \infty\}/\sim$, $p \in [1, \infty)$, measure space $(\Omega, \mathcal{F}, \mu)$.
- $L^\infty_\mu(\Omega) = \{\mathcal{F}\text{-m.b. } u\colon \Omega \to \mathbb{R} \mid \operatorname{esssup}_{\omega\in\Omega}|u(\omega)| < \infty\}/\sim$.
- $W^{k,p}(D) = \{u \in L^p(D) \mid \text{with weak derivatives}^3\ D^\alpha u \in L^p(D) \text{ for } |\alpha| \leq k\}$.
- $\mathcal{C}(\bar{D}) = \{u\colon \bar{D} \to \mathbb{R} \mid u \text{ is continuous}\}$ with $\bar{D} \subset \mathbb{R}^d$ closed and bounded ($\|u\|_{\mathcal{C}(\bar{D})} \coloneqq \sup_{x\in\bar{D}} |u(x)|$).
- Hilbert space $(H, (\cdot, \cdot)_H)$ with norm $\|u\|_H \coloneqq \sqrt{(u, u)_H}$.
- Space $\mathcal{L}(X, Y)$ of all bounded linear operators from X to Y (Banach spaces).

[3] $D^\alpha u \coloneqq v \in L^1_{\mathrm{loc}}(D)$ is the weak derivative of $u \in L^1_{\mathrm{loc}}(D)$ if

$$\int_D v\,\varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_D u D^\alpha \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty(D).$$

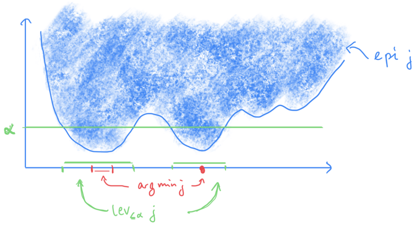
Notions that carry over from finite dimensions

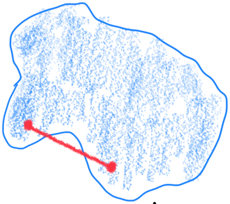
- Definitions of epi $\boldsymbol{j}$, lev$_{\leq\alpha}$ $\boldsymbol{j}$, argmin $\boldsymbol{j}$ for $j\colon X \to \mathbb{R} \cup \{\infty\}$.
- $C \subset X$ is **convex** if $u, v \in C$ implies

$$\lambda u + (1-\lambda)v \in C \quad (\forall \lambda \in [0,1]).$$

- $j\colon X \to \mathbb{R} \cup \{\infty\}$ **μ-strongly convex** ($\mu > 0$) or **convex** ($\mu = 0$) on $C \subset X$ if

$$\mu \frac{\lambda(1-\lambda)}{2}\|u-v\|^2 + j(\lambda u + (1-\lambda)v) \leq \lambda j(u) + (1-\lambda) j(v) \quad (\forall u, v \in C \text{ and } \lambda \in [0,1]).$$





A nonconvex set.

Dual space

The topological **dual space** of Banach space X is defined by $\boldsymbol{X^* := \mathcal{L}(X, \mathbb{R})}$ with norm

$$\|x^*\|_{X^*} := \sup_{\|x\|_X = 1} \{x^*(x) =: \langle x^*, x\rangle_{X^*, X}\}.$$

Example (Dual space of L^p_μ)

$L^p_\mu(\Omega)^*$ can be identified with $L^q_\mu(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p \in [1, \infty)$, via

$$\langle u^*, u\rangle_{(L^p_\mu)^*, L^p_\mu} := \int_\Omega u(\omega)v(\omega)\, \mathrm{d}\mu(\omega), \qquad u \in L^p_\mu, v \in L^q_\mu, u^* \in (L^p_\mu)^*.$$

Warning: $L^1_\mu(\Omega)$ is only a subspace of $L^\infty_\mu(\Omega)^*$!

Example (Bidual)

The topological bidual space is the dual of the dual space X^*; i.e., it is defined by $\boldsymbol{X^{**}} := \mathcal{L}(X^*, \mathbb{R})$. X can be identified with a closed subspace of X^{**} via

$$\langle x^{**}, x^*\rangle := \langle x^*, x\rangle \quad \forall x^* \in X^*.$$

Riesz representation theorem

Theorem (Riesz representation for Hilbert space H)

$H^* \cong H$, i.e., for every $u^* \in H^*$ there exists a unique $v \in H$ such that

$$\langle u^*, u\rangle_{H^*,H} = (v, u)_H \quad \forall u \in H, \quad \|u^*\|_{H^*} = \|v\|_H. \tag{1}$$

Conversely, for every $v \in H$, the linear functional u^* defined by (1) is in H^*.

Consequence: Hilbert spaces are reflexive. [4]

[4] A Banach space X is called **reflexive** if $X \ni x \mapsto \langle \cdot, x\rangle_{X^*,X} \in X^{**}$ is surjective, i.e.,

$$\forall x^{**} \in X^{**}\ \exists x \in X : \quad \langle x^{**}, x^*\rangle_{X^{**},X^*} = \langle x^*, x\rangle_{X^*,X} \quad \forall x^* \in X^*.$$

Derivatives, gradients

A function $F\colon O \subset X \to Y$ ($O \neq \emptyset$ open and X, Y Banach spaces) is **Gâteaux differentiable** if the limit

$$dF(x,h) = \lim_{t \to 0^+} \frac{F(x+th) - F(x)}{t} \in Y$$

exists for all $h \in X$ and $DF(x)\colon X \ni h \mapsto dF(x,h) \in Y$ is bounded and linear, i.e., $DF(x) \in \mathcal{L}(X,Y)$.

It is **Fréchet differentiable** if it also holds that

$$\|F(x+h) - F(x) - DF(x)h\|_Y = o(\|h\|_X) \quad \text{for } \|h\|_X \to 0.$$

Special case for $j\colon X \to \mathbb{R}$, X Hilbert space: the **gradient** $\nabla j\colon X \to X$ is the Riesz representation of Dj, i.e.,

$$(\nabla j(x), v)_X = \langle Dj(x), v\rangle_{X^*,X} \quad \forall v \in X.$$

Separability

A Banach space X is **separable** if it contains a countable dense subset, meaning there exists $Y = \{x_i \in X \mid i \in \mathbb{N}\} \subset X$ such that

$$\forall x \in X \, \forall \varepsilon > 0 \, \exists y \in Y : \quad \|x - y\|_X < \varepsilon.$$

Example

- $\mathcal{C}(\bar{D})$ with $\bar{D} \subset \mathbb{R}^d$ closed and bounded.
  (Polynomials with rational coefficients are dense by Weierstraß's approximation theorem.)
- $L^p(D)$, $W^{k,p}(D)$, $p \in [1, \infty)$, $D \subset \mathbb{R}^d$.
- X is separable if X^* is; (norm)-closed subspaces of a separable X are separable.

Not separable: $L^\infty(D)$, $W^{k,\infty}(D)$!

Strong, weak, weak* topologies

Topologies of interest on Banach space $(X, \|\cdot\|)$ with $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{X^*,X}$:

- **Strong topology** τ_s (generated on X by norm).
  x_n **strongly converges** to $\bar{x}$ ($x_n \to \bar{x}$) if
  $$\|x_n - \bar{x}\| \to 0.$$
- **Weak topology** τ_w (coarsest one on X so $\langle x^*, \cdot\rangle \colon X \to \mathbb{R}$ continuous $\forall x^* \in X^*$).
  x_n **weakly converges** to $\bar{x}$ ($x_n \rightharpoonup \bar{x}$) if
  $$\langle x^*, x_n\rangle \to \langle x^*, \bar{x}\rangle \quad \forall x^* \in X^*.$$
- **Weak* topology** τ_{w^*} (coarsest one on X^* so $\langle\cdot, x\rangle \colon X^* \to \mathbb{R}$ continuous $\forall x \in X$).
  x_n^* **weakly* converges** to $\bar{x}^*$ ($x_n^* \rightharpoonup^* \bar{x}^*$) if
  $$\langle x_n^*, x\rangle \to \langle \bar{x}^*, x\rangle \quad \forall x \in X.$$

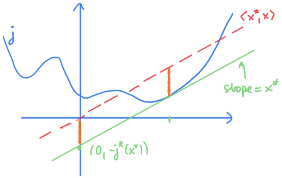
Hierarchy: $\tau_{w^*} \subset \tau_w \subset \tau_s$.

Closedness, lower semicontinuity, and compactness need to be qualified w.r.t. topology!

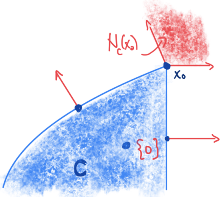
Notation: $\overline{C}^{\tau}$ ... closure of set C w.r.t. topology τ

τ**-lsc** ... $x_n \to^{\tau} x$ implies $\liminf_{n\to\infty} j(x_n) \geq j(x)$.

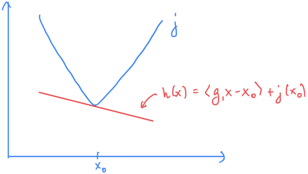
Notions defined with respect to the dual pairing



Convex conjugate:
$j^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - j(x)\}$.



Normal cone:
$N_C(x_0) := \{y \in X^* \mid \langle y, x - x_0 \rangle \leq 0 \, \forall x \in C\}$.



Subdifferential:

$\partial j(x_0) \coloneqq \{g \in X^* \mid j(x) \geq j(x_0) + \langle g, x - x_0 \rangle \, \forall x \in X\}$.

Compactness

Fact: closed and bounded sets are not compact in infinite dimensions.

Remedy: work with weaker topologies.

Theorem (Banach–Alaoglu)

Closed unit ball of X^* is weakly compact.*

Corollary

*Closed unit ball of **reflexive** X is weakly compact.*

Topological minimization theorem

Theorem

Let (X,τ) be a topological space and assume $j\colon X \to \mathbb{R} \cup \{\infty\}$ is τ-lower semicontinuous and such that $\mathrm{lev}_\alpha j$ is τ-compact. Then, $\inf_{x\in X} j(x) > -\infty$ and there exists some $\bar{x} \in X$ such that

$$j(\bar{x}) \le j(x) \quad \forall x \in X.$$

Proof. See Theorem 3.2.2 [5].

[5] Attouch, Buttazzo, Michaille. **Variational Analysis in Sobolev and BV Spaces** (2014).

Choice of topology

Choice of topology requires **give-and-take**: if τ_1 is stronger than τ_2, then

$$\overline{\mathrm{lev}_\alpha j}^{\tau_1} \quad \tau_1\text{-compact} \quad \Rightarrow \quad \overline{\mathrm{lev}_\alpha j}^{\tau_2} \quad \tau_2\text{-compact}$$

but

$$j \quad \tau_2\text{-lsc} \quad \Rightarrow \quad j \quad \tau_1\text{-lsc}.$$

Convexity, closedness, and lower semicontinuity

Proposition

(Strongly) ***closed convex*** *subsets of X are weakly sequentially closed.*

Corollary

Any ***continuous convex*** *functional $j \colon X \to \mathbb{R}$ is weakly lower semicontinous (lsc), i.e.,*

$$x_n \rightharpoonup x \quad \Rightarrow \quad \liminf_{n \to \infty} j(x_n) \geq j(x).$$

Proof. Use j τ-lsc $\quad\Leftrightarrow\quad$ epi j τ-closed $\quad\Leftrightarrow\quad$ $\mathrm{lev}_\alpha j$ τ-closed ($\alpha \in \mathbb{R}$).

Operators

A **compact operator** $A \in \mathcal{L}(X, Y)$ maps bounded sets to relatively bounded sets and has the property that

$$X \ni x_n \rightharpoonup x \quad \Rightarrow \quad Ax_n \to Ax \in Y \quad \text{(weak-to-strong continuity)}.$$

Example (Compactness via Sobolev embedding theorem)

Let $D \subset \mathbb{R}^d$ open bounded with Lipschitz boundary [6].

- The embedding $\iota\colon W^{k_1,p_1}(D) \to W^{k_2,p_2}(D)$ is continuous and compact if

$$k_1 - \frac{d}{p_1} > k_2 - \frac{d}{p_2} \quad \text{and} \quad k_1 > k_2.$$

- The embedding $\iota\colon W^{k,p}(D) \to \mathcal{C}^{\ell,\alpha}(\bar{D})$ is continuous and compact if

$$k - \frac{d}{p} > \ell + \alpha.$$

[6] meaning boundary can be represented as the graph of a Lipschitz continuous function.

Dual operator

Given an operator $A \in \mathcal{L}(X, Y)$, the **dual operator** $A^* \in \mathcal{L}(Y^*, X^*)$ is defined by

$$\langle A^* u, v \rangle_{X^*,X} = \langle u, Av \rangle_{Y^*,Y} \quad \forall u \in Y^*, \forall v \in X.$$

Theorem (Schauder)

$A \in \mathcal{L}(X, Y)$ is compact $\quad \Leftrightarrow \quad$ *$A^* \in \mathcal{L}(Y^*, X^*)$ is compact.*

Now let's mix in some stochasticity....

From now on let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space.

How about **measurability** in ∞-dimensions?

Vector-valued mappings

Banach space $(X, \|\cdot\|)$ with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{X^*,X}$.

A vector-valued mapping $x \colon \Omega \to X$ is said to be

- **strongly measurable**, if there exists a sequence of $\mathbb{P}$-simple functions [7] $\{x_n\}$ such that $\|x_n - x\| \to 0$ a.s. ($\mathbb{P}$-a.e.)
- **weakly measurable**, if $\omega \mapsto \langle x^*, x(\omega) \rangle$ is measurable for all $x^* \in X^*$.

A mapping $x^* \colon \Omega \to X^*$ is said to be

- **weakly* measurable**, if $\omega \mapsto \langle x^*(\omega), x \rangle$ is measurable for all $x \in X$.

Theorem (Pettis measurability)

x strongly measurable $\quad\Leftrightarrow\quad$ x separably-valued and weakly measurable.

[7] there exist $\chi_j \in X$, $F_j \in \mathcal{F}$ ($j = 1, \ldots, N$) such that

$$x_n(\omega) = \sum_{j=1}^N \chi_j \mathbf{1}_{F_j}(\omega), \quad \omega \in \Omega.$$

Bochner space and its dual

The **Bochner space** $L^p_{\mathbb{P}}(\Omega, X)$ is the set of all (equivalence classes of) strongly measurable $x\colon \Omega \to X$ having finite norm given by

$$\|x\|_{L^p_{\mathbb{P}}(\Omega,X)} := \begin{cases} \left(\int_\Omega \|x(\omega)\|_X^p \, \mathrm{d}\mathbb{P}(\omega)\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{\omega\in\Omega} \|x(\omega)\|_X, & p = \infty. \end{cases}$$

The limit of the integrals of $\mathbb{P}$-simple functions x_n gives the **Bochner integral**

$$\mathbb{E}[x] := \int_\Omega x(\omega) \, \mathrm{d}\mathbb{P}(\omega) = \lim_{n\to\infty} \int_\Omega x_n(\omega) \, \mathrm{d}\mathbb{P}(\omega) \in X.$$

Theorem

Suppose X is reflexive or X^* is separable. Then for all $1 \le p < \infty$, we have the isometric isomorphism

$$(L^p_{\mathbb{P}}(\Omega, X))^* = L^q_{\mathbb{P}}(\Omega, X^*), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. See, e.g., Section 1.3 in [8]

[8] Hytönen et al. **Analysis in Banach spaces** (2016).

Implicit measurability theorem

A helpful result for showing the **measurability of feasible points**:

Theorem (Filippov)

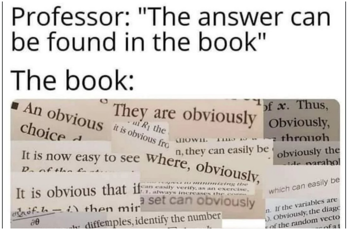
Let $F\colon X \times \Omega \to Y$ be a Carathéodory mapping, and suppose $C(\omega) \subset X$ and $D(\omega) \subset Y$ are closed sets that depend measurably on ω. Then the set

$$E = \{\omega \in \Omega \mid \exists x \in C(\omega) \quad \text{such that} \quad F(x,\omega) \in D(\omega)\}$$

is measurable, and there exists a measurable function $x\colon E \to X$ such that

$$x(\omega) \in C(\omega) \quad \text{and} \quad F(x(\omega),\omega) \in D(\omega) \quad \forall \omega \in E.$$

Proof. See Section 8.1 [9] for the proof and other measurability properties.



[9] Aubin, Frankowska. **Set-valued analysis** (1990).

Random linear operators

A random linear operator $A\colon \Omega \to \mathcal{L}(Y, W)$ is called **strongly measurable** if for all $y \in Y$, $\omega \mapsto A(\omega)y$ is strongly measurable.

Theorem (Hans [10])

Let $A : \Omega \to \mathcal{L}(Y, W)$. Then

- *$A(\omega)$ is invertible a.s. if and only if $\operatorname{ran}(A^*(\omega)) = Y^*$ a.s.*
- *if $A(\omega)$ is invertible a.s., then $A^*(\omega)$ is invertible and $(A^*(\omega))^{-1} = (A^{-1}(\omega))^*$,*
- *if any of the operators $A(\omega)$, $A^{-1}(\omega)$, $A^*(\omega)$, $(A^{-1}(\omega))^*$ is strongly measurable, then they all are.*

[10] Hans. **Inverse and adjoint transforms of linear bounded random transforms** (1957).

Derivative of expectation

Lemma

Let $J\colon X \times \Omega \to \mathbb{R}$. *Suppose*

1. $j(v) = \mathbb{E}[J(v,\cdot)]$ *is well-defined and finite-valued for all* $v \in O \subset X$ *(*O *open),*
2. $J_\omega := J(\cdot,\omega)$ *is a.s. Fréchet differentiable at* $x \in O$*,*
3. *there exists a positive random variable* $C \in L^1_{\mathbb{P}}(\Omega)$ *such that for all* $v \in O$ *and almost every* $\omega \in \Omega$*,*
$$\|DJ_\omega(v)\|_{X^*} \le C(\omega).$$

Then j *is Fréchet differentiable at* u *and*
$$Dj(x) = \mathbb{E}[DJ_\omega(x)].$$

Proof. See, e.g, Lemma C.3 in [11].

[11] G., Scarinci. **Stochastic proximal gradient methods for nonconvex problems in Hilbert spaces** (2021).

Today's agenda

Two-stage problems

Two-stage ("recourse") problem from stochastic programming (on Banach spaces U, Y):

$$\min_{u\in U_{\text{ad}}} \quad J_1(u) + \mathbb{E}\left[\min_{y\in Y} J_2(u, y, \cdot)\right] \quad \text{s.t.} \quad y \in Y_{\text{ad}}(u,\omega) \quad \text{a.s.}$$

Sequence of events:

$$\underbrace{u}_{\substack{\text{first-stage decision}\\ \text{(control)}}} \rightarrow \underbrace{\omega}_{\text{random event}} \rightarrow \underbrace{y}_{\substack{\text{second-stage decision}\\ \text{(state)}}}$$

Interchangeability principle (for J_2 normal integrand [12]):

$$\min_{y(\cdot)\in L^p_{\mathbb{P}}(\Omega,Y)} \mathbb{E}[J_2(u, y(\cdot), \cdot)] = \mathbb{E}\left[\min_{y\in Y} J_2(u, y, \cdot)\right].$$

[12] meaning $(\text{epi } J_2)(\omega) := \{(u, y, \alpha) \in U \times Y \times \mathbb{R} \mid J_2(u, y, \omega) \leq \alpha\}$ is closed-valued and measurable.

Recourse structures in stochastic programming

Definition (Recourse structures)

The problem

$$\min_{u \in U_{\mathrm{ad}}} \quad J_1(u) + \mathbb{E}\left[\min_{y \in Y} J_2(u, y, \cdot)\right] \quad \text{s.t.} \quad y \in Y_{\mathrm{ad}}(u, \omega) \quad \text{a.s.}$$

is said to satisfy the assumption of

- **complete recourse**, if for every u there exists a feasible $y(\omega)$ $\mathbb{P}$-a.e. ω;
- **relatively complete recourse**, if for every *feasible* u there exists a feasible $y(\omega)$ $\mathbb{P}$-a.e. ω.
- ... other notions (fixed, simple) not used here...

Relevance: special role in optimality conditions (two-stage problems).

Optimality conditions for constrained problem

Deterministic setting

Theorem

Let $U_{\text{ad}} \subset U$ be convex and j be Gâteaux differentiable on an open set covering U_{ad}. Then a necessary condition for $\bar{u} \in U_{\text{ad}}$ to be a solution to $\min_{u \in U_{\text{ad}}} j(u)$ is

$$dj(\bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in U_{\text{ad}}. \tag{2}$$

In case j is convex, (2) is a sufficient condition for optimality of $\bar{u}$.

Proof. **Necessity**: For any $v \in U_{\text{ad}}$ and $\lambda \in (0, 1]$,

$$v(\lambda) = \bar{u} + \lambda(v - \bar{u}) \in U_{\text{ad}}.$$

Since $\bar{u}$ is optimal, $j(v(\lambda)) \geq j(\bar{u})$, and also

$$\frac{1}{\lambda}(j(v(\lambda)) - j(\bar{u})) \geq 0.$$

Taking the limit as $\lambda \to 0$, we obtain (2).

Sufficiency: follows from $j(v) - j(\bar{u}) \geq dj(\bar{u})(v - \bar{u})$.

Problems with reducible objective

Framework for equality constraints

$$\min_{(u,y)\in U_{\mathrm{ad}}\times L^p_{\mathbb{P}}(\Omega,Y)} \quad J_1(u)+\mathbb{E}[J_2(u,y(\cdot),\cdot)]$$
$$\text{s.t.} \quad e(u,y(\omega),\omega)=0 \quad \text{a.s.}$$

- Control problems are often modeled so that a **control-to-state** operator $S\colon U\times\Omega\to Y$ is well-defined (e.g., through PDE theory).
- Problem can be **reduced to single-stage** form
$$\min_{u\in U_{\mathrm{ad}}} \quad \{j(u):=J_1(u)+\mathbb{E}[J_2(u,S(u,\cdot),\cdot)]\}.$$
- For optimality conditions, we need $Dj(u)$ and therefore also the derivative $D_uS(u,\omega)$ $\rightsquigarrow$ apply implicit function theorem.

Notation: $\boldsymbol{J_{2,\omega}(u,y(\omega))}=J_2(u,y(\omega),\omega)$, $\boldsymbol{S_\omega(u)}=S(u,\omega)$, $\boldsymbol{e_\omega(u,y(\omega))}=e(u,y(\omega),\omega)$.

Implicit function theorem

Theorem (IFT)

Let U, Y, Z be Banach spaces, $e\colon O \to Z$ (open $O \subset U \times Y$) continuously Fréchet differentiable, $(\bar{u}, \bar{y}) \in O$ such that $e(\bar{u}, \bar{y}) = 0$ and $D_y e(\bar{u}, \bar{y}) \in \mathcal{L}(Y, Z)$ has a bounded inverse.

Then there exists an open neighborhood $\mathcal{N}_U(\bar{u}) \times \mathcal{N}_Y(\bar{y}) \subset O$ of $(\bar{u}, \bar{y})$ and a unique continuous function $S\colon \mathcal{N}_U(\bar{u}) \to Y$ such that

1. $S(\bar{u}) = \bar{y}$,
2. *For all $u \in \mathcal{N}_U(\bar{u})$, there exists exactly one $y \in \mathcal{N}_Y(\bar{y})$ with $e(u, y) = 0$, namely $y = S(u)$.*

Moreover, $S\colon \mathcal{N}_U(\bar{u}) \to Y$ is continuously Fréchet differentiable with

$$DS(u) = -D_y e(u, S(u))^{-1} D_u e(u, S(u)).$$

If $e\colon O \to Z$ is m-times continuously Fréchet differentiable, then so is $S\colon \mathcal{N}_U(\bar{u}) \to Y$.

Adjoint method

Approach for computing derivative of $\hat{J}_\omega(u) := J_1(u) + J_{2,\omega}(u, S_\omega(u))$

- **Chain rule + adjoint operator**:

$$\begin{aligned}&\langle D\hat{J}_\omega(u), h\rangle_{U^*,U}\\ &= \langle DJ_1(u), h\rangle_{U^*,U} + \langle D_u J_{2,\omega}(u, S_\omega(u)), h\rangle_{U^*,U} + \langle D_y J_{2,\omega}(u, S_\omega(u)), DS_\omega(u)h\rangle_{Y^*,Y}\\ &= \langle DJ_1(u), h\rangle_{U^*,U} + \langle D_u J_{2,\omega}(u, S_\omega(u)), h\rangle_{U^*,U} + \langle (DS_\omega(u))^* D_y J_{2,\omega}(u, S_\omega(u)), h\rangle_{U^*,U}.\end{aligned}$$

- **IFT in adjoint form**: $(DS_\omega(u))^* = -(D_u e_\omega(u, S_\omega(u)))^* D_y e_\omega(u, S_\omega(u))^{-*}$.

$$\Rightarrow \quad DS_\omega(u))^* D_y J_{2,\omega}(u, S_\omega(u)) = -(D_u e_\omega(u, S_\omega(u)))^* \underbrace{D_y e_\omega(u, S_\omega(u))^{-*} D_y J_{2,\omega}(u, S_\omega(u))}_{=: -p_\omega}$$

- **Derivative**:

$$D\hat{J}_\omega(u) = DJ_1(u) + D_u J_{2,\omega}(u, S_\omega(u)) + D_u e_\omega(u, S_\omega(u)))^* p_\omega,$$

where p_ω solves the **adjoint equation**

$$D_y e_\omega(u, S_\omega(u))^* p_\omega = -D_y J_{2,\omega}(u, S_\omega(u)).$$

Alternative approach: sensivity method; see section 1.6 in Hinze, Pinnau, Ulbrich, Ulbrich. **Optimization with PDE constraints** (2009).

Optimality conditions for reducible objective

Summary: necessary optimality conditions for $\bar{u} \in U_{\text{ad}}$ to be a solution to

$$\min_{(u,y)\in U_{\text{ad}}\times L^p_{\mathbb{P}}(\Omega,Y)} \quad J_1(u) + \mathbb{E}[J_2(u,y(\cdot),\cdot)]$$
$$\text{s.t.} \quad e(u,y(\omega),\omega) = 0 \quad \text{a.s.}$$

given by

$$\langle DJ_1(\bar{u}) + \mathbb{E}[D_u J_{2,\omega}(\bar{u}, S_\omega(\bar{u})) + D_u e_\omega(\bar{u}, S_\omega(\bar{u}))^* \bar{p}_\omega], v - \bar{u}\rangle_{U^*,U} \geq 0 \quad \forall v \in U_{\text{ad}},$$

where $\bar{p}_\omega$ solves the adjoint equation

$$D_y e_\omega(\bar{u}, S_\omega(\bar{u}))^* p_\omega = -D_y J_{2,\omega}(\bar{u}, S_\omega(\bar{u})).$$

Note: "$D\mathbb{E}[J_\omega] = \mathbb{E}[DJ_\omega]$" needs to be justified above.

Problems without reducible form

Historical development

PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 1, 1976

STOCHASTIC CONVEX PROGRAMMING:
BASIC DUALITY

R. T. ROCKAFELLAR AND R. J.-B. WETS

A duality theory is developed for stochastic programs with convex objective and convex constraints. The problem consists in selecting $x_1 \in R^{n_1}$ and $x_2 \in \mathscr{L}^\infty(S, \Sigma, \sigma; R^{n_2})$ so as to satisfy the constraints and minimize total expected cost, where σ is a probability measure and the constraints as well as the objective are functions of the random elements of the problem. Under the additional restriction that x_1 and $x_2(s)$ belong to compact subsets of R^{n_1} and R^{n_2} respectively, it is shown that the problem is equivalent to the more common dynamic formulation for stochastic programs with recourse, a basic duality theorem — of the type $\min = \sup$ — is proved and qualitative results on the existence of dual solutions are derived.

1. Introduction. In this paper we study a two-stage stochastic optimization problem associated with the following heuristic model. First, a vector x_1 in R^{n_1} is chosen subject to the constraints

(1.1) $\quad x_1 \in C_1 \quad \text{and} \quad f_{1i}(x_1) \leqq 0, \qquad i = 1, \cdots, m_1,$

at a cost represented by the expression $f_{10}(x_1)$. Next an element s of S is "observed", where (S, Σ, σ) is a probability space. Finally, a vector $x_2(s)$ in R^{n_2} is chosen subject to the constraints

(1.2) $\quad x_2(s) \in C_2 \quad \text{and} \quad f_{2i}(s, x_1, x_2(s)) \leqq 0, \qquad i = 1, \cdots, m_2,$

at a cost $f_{20}(s, x_1, x_2(s))$. The problem is to choose x_1 and the function $x_2(\cdot)$ so as to minimize the total expected cost

(1.3) $\quad f_{10}(x_1) + \int_S f_{20}(s, x_1, x_2(s))\, \sigma(ds).$

This is a *stochastic programming problem with recourse*; the function $x_2(\cdot)$ specifies the recourse decision.

KKT conditions [13]

- Choice of function space $L^\infty_{\mathbb{P}}(\Omega, \mathbb{R}^n)$.
- Lagrange multipliers in $L^1_{\mathbb{P}}(\Omega, \mathbb{R}^n)$ under **relatively complete recourse** condition, **constraint qualification**.

  $\rightsquigarrow$ convenient for algorithm building!

[13] Rockafellar, Wets. **Stochastic convex programming: Kuhn-Tucker conditions** (1975), **Stochastic convex programming: basic duality** (1976), **Stochastic convex programming: relatively complete recourse and induced feasibility** (1976), **Stochastic convex programming: singular multipliers and extended duality singular multipliers and duality** (1976).

Problems without reducible form

Generalized Rockafellar/Wets framework [14]

$$\min_{(u,y)\in U_{\text{ad}}\times L^p_{\mathbb{P}}(\Omega,Y)} \quad \{\mathcal{J}(u,y) := J_1(u) + \mathbb{E}[J_2(u,y(\cdot),\cdot)]\}$$

$$\text{s.t.} \quad e(u,y(\omega),\omega) = 0 \quad \text{a.s.}, \quad \boldsymbol{i(u,y(\omega),\omega) \leq_K 0} \quad \text{a.s.}$$

Existence of Lagrange multipliers under **constraint qualification** (strict feasibility):

$$0 \in \operatorname{int}\operatorname{dom} v,$$

with value function of *perturbed problem*

$$v(\phi) := \inf_{(u,y)\in U\times L^p_{\mathbb{P}}(\Omega,Y)} \mathcal{J}(u,y) + \delta_{F_{\text{ad},\phi}}(u,y),$$

$$\text{where} \quad F_{\text{ad},\phi} := \{(u,y) \in U_{\text{ad}} \times L^p_{\mathbb{P}}(\Omega,Y) \mid \boldsymbol{e(u,y(\omega),\omega) = \phi_e(\omega)} \text{ a.s.}, \ \boldsymbol{i(u,y(\omega),\omega) \leq_K \phi_i(\omega)} \text{ a.s.}\}.$$

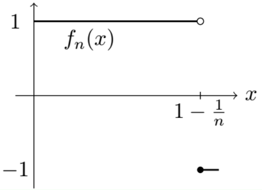
Notation: $k \leq_K 0 \quad \Leftrightarrow \quad -k \in K$.

[14] G., Wollner. **Optimality conditions for almost sure state constraints** (2021). G., Hintermüller. **Moreau–Yosida regularization for almost sure state constraints** (2022). G., Henrion. **Optimality conditions with chance constraints** (2024).

Choice of function spaces depends on CQ

Fact: for $p \in [1, \infty)$, $K \subset L^p_{\mathbb{P}}(\Omega)$ has an empty interior unless Ω is finite dimensional.

Example ($L^2_+(0,1) = \{f \in L^2(0,1) \mid f \geq 0\}$ has empty interior [15])



For $\mathbb{P}$ = Lebesgue measure; $\Omega = (0,1)$:
$f \equiv 1$ is not an interior point of $K = L^2_+(0,1)$: for

$$f_n(x) = \begin{cases} 1, & [0, 1 - \frac{1}{n}) \\ -1, & [1 - \frac{1}{n}, 1] \end{cases},$$

we have $\|f_n - f\|_{L^2(0,1)} \to 0$ but $f_n \notin K$ for all n.
$\Rightarrow f \notin \operatorname{int} K$.

Consequence: CQ requires using function spaces with sets having nonempty interiors ($L^\infty_{\mathbb{P}}(\Omega, X)$, X sufficiently regular).

[15] Tröltzsch. **Optimal control of partial differential equations** (2009).

Yosida-Hewitt-type decomposition on $L^\infty_{\mathbb{P}}(\Omega, X)^*$

For X separable

Theorem ([16])

Every $v^* \in L^\infty_{\mathbb{P}}(\Omega, X)^*$ has a unique decomposition

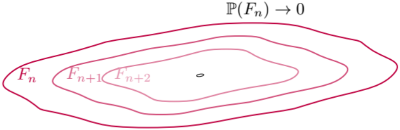
$$v^* = v + v^\circ,$$

where v is absolutely continuous* and v° is singular** relative to $\mathbb{P}$.

($\star$) $\exists$ weakly* measurable $x^* : \Omega \to X^*$ with $\|x^*(\cdot)\|_{X^*} \in L^1_{\mathbb{P}}(\Omega)$ s.t.

$$v(x) = \int_\Omega \langle x^*(\omega), x(\omega)\rangle_{X^*,X} \, \mathrm{d}\mathbb{P}(\omega).$$

($\star\star$) There exist $\{F_n\} \subset \mathcal{F}$ such that
$\forall n : x|_{F_n} = 0 \Rightarrow \langle \lambda^\circ, x\rangle = 0.$



If X **separable and reflexive**, absolutely continuous functionals belong to $\boldsymbol{L^1_{\mathbb{P}}(\Omega, X^*)}$.

[16] Ioffe, Levin. **Subdifferentials of convex functions** (1972).
Corrected proof in Levin. **The Lebesgue decomposition for functionals on the vector-function space L^∞_X (1974).**

Existence of Lagrange multipliers

Convex problems: conjugate duality à la Rockafellar and Wets:

1. Show saddle points $\lambda^* := (\lambda_e^*, \lambda_i^*) \in (L^p_{\mathbb{P}}(\Omega, W))^* \times (L^p_{\mathbb{P}}(\Omega, R))^*$ of Lagrangian

$$\bar{L}(u, y, \lambda^*) = j(u, y) + \langle \lambda_e^*, e(u, y, \cdot) \rangle + \langle \lambda_i^*, i(u, y, \cdot) \rangle$$

   exist (assumption: CQ + bounded feasible set or coercive j).

2. Use Yosida–Hewitt decomposition

$$\lambda_e^* = \lambda_e^a + \lambda_e^\circ \in L^\infty_{\mathbb{P}}(\Omega, W)^*, \quad \lambda_i^* = \lambda_i^a + \lambda_i^\circ \in L^\infty_{\mathbb{P}}(\Omega, R)^*$$

   and argue that singular multipliers $\lambda_e^\circ, \lambda_i^\circ$ vanish (with **relatively complete recourse**).
   Generally **not** satisfied by optimal control problems with state constraints!

Nonconvex problems: existence of Lagrange multipliers with [17]

[17] Zowe, Kurcyusz. **Regularity and stability for the mathematical programming problem in Banach spaces** (1979).

Problem with purely singular multipliers

Example ([18])

Problem:

$$\min_{u\geq 0} -u \quad \text{s.t.} \quad G(u,\omega) \coloneqq u-\omega \leq 0 \text{ in } \Omega \coloneqq [1,2]. \tag{3}$$

- Solution $u^* = 1$.
- Optimality conditions:

$$\langle \lambda^*, \mathbf{1}(\cdot)\rangle = -1, \qquad G(u^*,\omega) \leq 0 \text{ in } \Omega, \qquad \lambda^* \in \mathcal{K}^-, \qquad \langle \lambda^*, G(u^*,\cdot)\rangle = 0.$$

- $F_n \coloneqq (1, 1+1/n)$ so that $\mathbb{P}(F_n) \to 0$ as $n \to \infty$ for Lebesgue measure $\mathbb{P}$.
- Since $G(u^*,\cdot) < -1/n$ on $\Omega \setminus F_n$, we have $B_{1/n}(G(u^*,\cdot)) \subset L^\infty_-(\Omega)$.
- $y \in L^\infty(\Omega)$ with $y = 0$ on F_n, ρ_n small enough so that $\rho_n y \in B_{1/n}(G(u^*,\cdot))$.
- Due to complementarity conditions, we have

$$\langle \lambda^*, G(u^*,\cdot) \pm \rho_n y\rangle = \pm\rho_n \langle \lambda^*, y\rangle \leq 0 \quad \Rightarrow \langle \lambda^*, y\rangle = 0 \quad \forall y : y = 0 \text{ on } F_n$$
$$\Rightarrow \lambda^* = \lambda^\circ \neq 0 \quad (\langle \lambda^*, \mathbf{1}(\cdot)\rangle = -1).$$

Notation: $\mathbf{1}(\omega) = 1$ for all $\omega \in \Omega$, $\quad \mathcal{K}^- = \{k^* \in L^\infty(\Omega)^* \mid \langle k^*, k\rangle \leq 0 \, \forall k \in L^\infty_-(\Omega)\}$.

[18] Bonnans. **Convex and stochastic optimization** (2019).

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3. Optimality conditions
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4. **Case study**
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5. Stochastic approximation
   - Results in Hilbert spaces, handling of numerical error.

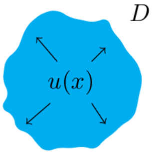
Case study

Optimal control of stationary heat source under uncertainty

$$\min_{(u,y)\in U_{\text{ad}}\times\mathcal{Y}} \frac{1}{2}\mathbb{E}\left[\|y-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2$$

$$\text{s.t.} \; -\nabla\cdot(a(x,\omega)\nabla y(x,\omega)) = u(x)+f(x,\omega), \quad x\in D \quad \text{a.s.,}$$

$$y(x,\omega)=0, \quad x\in\partial D \quad \text{a.s.} \qquad \text{(P)}$$



Theoretical questions:

- Solvability of random PDE
- Solvability of optimization problem
- Optimality conditions

Notation: $\nabla\cdot(a(x,\omega)\nabla y(x,\omega)) = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(a(x,\omega)\frac{\partial y(x,\omega)}{\partial x_i}\right)$

Case study
Solution to the random PDE

Lemma

Suppose $D \subset \mathbb{R}^d$ is open and bounded with Lipschitz boundary, $u, f(\cdot,\omega) \in U := L^2(D)$, and $0 < a_{\min} < a(x,\omega) < a_{\max} < \infty \; \forall x \in D$. Then ***there exists a unique weak solution $y(\cdot,\omega) \in H_0^1(D)$*** *to PDE in (P) that satisfies*

$$\int_D a(x,\omega)\nabla y(x,\omega) \cdot \nabla v(x) \; dx = \int_D (u(x) + f(x,\omega))v(x) \; dx \quad \forall v \in H_0^1(D).$$

Moreover, there exists $C_1 > 0$ such that

$$\|y(\cdot,\omega)\|_{H^1(D)} \leq C_1(\|u\|_{L^2(D)} + \|f(\cdot,\omega)\|_{L^2(D)}). \tag{4}$$

Moreover, if $a \in L^\infty_{\mathbb{P}}(\Omega, L^\infty(D)), f \in L^p_{\mathbb{P}}(\Omega, L^2(D))$, then ***$y \in \mathcal{Y} := L^p_{\mathbb{P}}(\Omega, H_0^1(D))$***.

Proof. Existence/uniqueness by Lax–Milgram lemma. Strong measurability by Filippov's theorem.

Notation: $H_0^1(D) := \overline{C_c^\infty(D)}^{\|\cdot\|_{H^1(D)}}$ and $H^1(D) = W^{1,2}(D)$.

Case study
Definition of control-to-state operator

Lemma justifies writing PDE as operator equation in $H^{-1}(D) := (H_0^1(D))^*$:

$$\boldsymbol{A(\omega)y(\omega) = B(u + f(\omega))}$$

with

$$A(\omega)\colon H_0^1(D) \to H^{-1}(D) \qquad A(\omega)y := -\nabla \cdot (a(\cdot,\omega)\nabla y),$$
$$B\colon L^2(D) \to H^{-1}(D) \qquad \text{(compact) embedding.}$$

(Parametrized) **control-to-state map**

$$S\colon \underbrace{U}_{:=L^2(D)} \times \Omega \to \underbrace{Y}_{:=H_0^1(D)}, \quad (u,\omega) \mapsto A^{-1}(\omega)B(u + f(\omega)).$$

Linear and continuous in u by (4)!

Case study

Existence of solution

Reduced formulation of problem (P):

$$\left.\begin{aligned}&\min_{(u,y)\in U_{\text{ad}}\times\mathcal{Y}} \frac{1}{2}\mathbb{E}\left[\|y-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2\\ &\text{s.t.}\quad A(\omega)y(\omega)=B(u+f(\omega))\end{aligned}\right\} \longrightarrow \min_{u\in U_{\text{ad}}} \frac{1}{2}\mathbb{E}\left[\|S(u,\cdot)-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2.$$

Proposition

Suppose $y_d \in L^2(D)$, $f \in L^2_{\mathbb{P}}(\Omega, L^2(D))$, and $\emptyset \neq U_{\text{ad}} \subset L^2(D)$ is closed and convex. Then if U_{ad} is bounded or $\mu > 0$, there exists a solution to problem (P).

Case study
Existence of solution

Proof.

- $j(u) := \frac{1}{2}\mathbb{E}\left[\|S(u,\cdot) - y_d\|_U^2\right] + \frac{\mu}{2}\|u\|_U^2$ bounded below $\Rightarrow j^* = \inf_{u \in U_{\text{ad}}} j(u)$ exists.
- Take minimizing sequence $\{u_n\} \subset U_{\text{ad}}$ with $j(u_n) \to j^*$ (bounded by assumption).
- $L^2(D)$ is reflexive, so there exists $\{u_{n_k}\}$ and $\bar{u} \in U_{\text{ad}}$ such that $u_{n_k} \rightharpoonup u^*$.
- j continuous since $u \mapsto \frac{1}{2}\|S(u,\omega) - y_d\|_U^2 + \frac{\mu}{2}\|u\|_U^2$ continuous a.s. and $S(u,\cdot) \in L^2_{\mathbb{P}}(\Omega, H_0^1(D))$.
- j is weakly lsc since it is continuous* and convex $\Rightarrow u^*$ solves problem (P).

Case study
Optimality conditions (1/2)

Necessary and sufficient conditions in an optimum $\bar{u}$:

$$\langle Dj(\bar{u}), v - \bar{u}\rangle_{U^*,U} \geq 0 \quad \forall v \in U_{\text{ad}}.$$

Computation using $D_u S(u,\omega) = D_u A^{-1}(\omega)B(u + f(\omega)) = A^{-1}(\omega)B$:

$$\begin{aligned}
\langle Dj(\bar{u}), v\rangle_{U^*,U} &= \left\langle D_u \left(\frac{1}{2}\mathbb{E}\left[\|S(\bar{u},\cdot) - y_d\|_U^2\right] + \frac{\mu}{2}\|\bar{u}\|_U^2\right), v\right\rangle_{U^*,U} \\
&= \int_\Omega \left\langle D_u \left(\frac{1}{2}\|S(\bar{u},\omega) - y_d\|_U^2\right), v\right\rangle_{U^*,U} \mathrm{d}\mathbb{P}(\omega) + \left\langle D_u \frac{\mu}{2}\|\bar{u}\|_U^2, v\right\rangle_{U^*,U} \\
&= \int_\Omega \langle S(\bar{u},\omega) - y_d, D_u S(\bar{u},\omega)v\rangle_{U^*,U} \, \mathrm{d}\mathbb{P}(\omega) + \langle \mu\bar{u}, v\rangle_{U^*,U} \\
&= \int_\Omega \langle D_u S(\bar{u},\omega)^*(S(\bar{u},\omega) - y_d), v\rangle_{U^*,U} \, \mathrm{d}\mathbb{P}(\omega) + \langle \mu\bar{u}, v\rangle_{U^*,U} \\
&= \int_\Omega \langle B^* \underbrace{A^{-*}(\omega)(S(\bar{u},\omega) - y_d)}_{=:\bar{p}(\omega)}, v\rangle_{U^*,U} \, \mathrm{d}\mathbb{P}(\omega) + \langle \mu\bar{u}, v\rangle_{U^*,U}.
\end{aligned}$$

$\rightsquigarrow$ **Introduction of adjoint variable $\bar{p}(\omega)$ allows for numerical computation.**

Case study

Optimality conditions (2/2)

Resulting optimality conditions:

$$\begin{aligned} \langle \mathbb{E}[B^*\bar{p}(\cdot)] + \mu\bar{u}, v - \bar{u}\rangle_{U^*,U} \geq 0 \quad \forall v \in U_{\mathrm{ad}}, \\ A^*(\omega)\bar{p}(\omega) = \bar{y}(\omega) - y_d \quad \text{a.s.} \\ A(\omega)\bar{y}(\omega) = B(\bar{u} - f(\omega)) \quad \text{a.s.} \end{aligned}$$

Or more explicitly:

$$\begin{aligned} &\langle \mathbb{E}[B^*\bar{p}(\cdot)] + \mu\bar{u}, v - \bar{u}\rangle_{U^*,U} \geq 0 \quad \forall v \in U_{\mathrm{ad}}, \\ &-\nabla \cdot (a(x,\omega)\nabla\bar{p}(x,\omega)) = \bar{y}(x,\omega) - y_d(x) \text{ on } D \times \Omega, \qquad \bar{p}(x,\omega) = 0 \text{ on } \partial D \times \Omega, \\ &-\nabla \cdot (a(x,\omega)\nabla\bar{y}(x,\omega)) = \bar{u}(x) - f(x,\omega) \text{ on } D \times \Omega, \qquad \bar{y}(x,\omega) = 0 \text{ on } \partial D \times \Omega. \end{aligned}$$

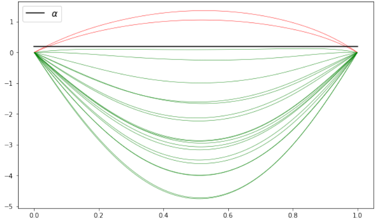
Gradient $\nabla j(\bar{u}) = \mathbb{E}[\bar{p}] + \mu\bar{u} \in L^2(D)$ (application of Riesz representation theorem).

The model problem with almost sure state constraints (irreducible)

$$\begin{aligned}
\min_{(u,y)\in U_{\text{ad}}\times\mathcal{Y}} \quad & \frac{1}{2}\mathbb{E}\left[\|y-y_d\|_U^2\right]+\frac{\mu}{2}\|u\|_U^2 \\
\text{s.t.} \quad -\nabla\cdot(a(x,\omega)\nabla y(x,\omega)) &= u(x)+f(x,\omega), \quad x\in D \quad \text{a.s.}, \\
y(x,\omega) &= 0, \quad x\in\partial D \quad \text{a.s.}, \\
\boldsymbol{y(x,\omega)\leq\alpha} & \quad x\in D \quad \text{a.s.}
\end{aligned} \qquad (\mathrm{P}_s)$$

- Problem (P_s) fits the framework of a two-stage decision problem.
- More forgiving model: **chance-constrained** setting

  $$\mathbb{P}(y(x,\omega)\leq\alpha\ \forall x\in D)\geq p, \quad p\in(0,1).$$
- Theoretical challenges (optimality conditions) and numerical methods currently unsatisfactory.



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Solution of stochastic optimization problems

Strategies for solving stochastic optimization problem

$$\min_{u \in U} \quad \left\{ j(u) = \mathbb{E}[J(u,\xi)] := \int_\Omega J(u,\xi(\omega))\, \mathrm{d}\mathbb{P}(\omega) \right\} :$$

- For small stochastic dimension: quadrature/other discretization for the integral.
- Sample average approximation (SAA): take one-time sample $\{\hat{\xi}^1, \dots, \hat{\xi}^m\}$ and solve

  $$\min_{u \in U} \quad \frac{1}{m} \sum_{i=1}^{m} J(u, \hat{\xi}^i)$$

  using, e.g., deterministic optimization method.
- “Stochastic approximation” (SA): dynamically sample while optimizing, e.g., with stochastic gradient method (SG)

  $$u^{n+1} := u^n - t_n G(u^n, \xi^n), \quad G(u^n, \xi^n) \approx \nabla j(u^n).$$

Inspiration for SA: up-and-down method

Dixon and Mood's "up-and-down" method [19] for finding the root of a function, demonstrated on an application where the critical height for explosives to detonate is determined experimentally:

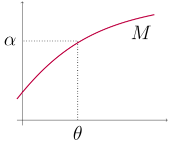
RECORD OF A SAMPLE OF SIXTY TESTS

Normalized Height		Number of x's	Number of o's
2.0	x	1	
1.7	x x x x x x x x x x	10	
1.4	o x o x x x x x o o x o x x x o x x x o x x o x x o x	18	9
1.1	o x o o o o o o o o o o o o o o o o x o	2	18
0.8	o o		2

FIGURE 1

[19] Dixon and Mood. **A method for obtaining and analyzing sensitivity data** (1948).

Stochastic approximation (origins)



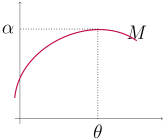
- Stochastic version by Robbins & Monro [20] for computing the (unique) root of an equation $M(u) = \alpha$, for a monotone function $M(u) = \int_{-\infty}^{\infty} y \, \mathrm{d}H(y|u)$.
- Iterations of the form

$$u^{n+1} = u^n - t_n(y^n - \alpha),$$

$y^n \ldots$ random variable with CDF $H(y|u^n)$.

- Kiefer & Wolfowitz [21] procedure for maximizing a regression function M using central-difference approximations:

$$u^{n+1} = u^n - t_n \frac{y^{2n-1} - y^{2n}}{c_n}.$$



Convergence in L^2 and in probability with proper choice of t_n, c_n.

[20] Robbins, Monro. **A stochastic approximation method** (1951).

[21] Kiefer, Wolfowitz. **Stochastic estimation of the maximum of a regression function** (1952).

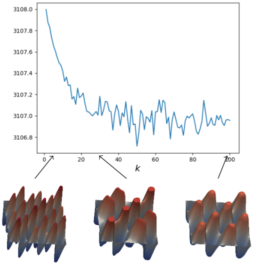
SA in function spaces

Literature:

- Goldstein. **Minimizing noisy functionals in Hilbert space: an extension of the Kiefer–Wolfowitz procedure** (1988).
- Yin and Zhu. **On *H*-valued Robbins–Monro processes** (1990).
- Culioli and Cohen. **Decomposition/coordination algorithms in stochastic optimization** (1990).
- Barty, Roy, and Strugarek. **Hilbert-valued perturbed subgradient algorithms** (2007).
- Bittar, Carpentier, Chancelier. **The stochastic auxiliary problem principle in Banach spaces: measurability and convergence** (2022).

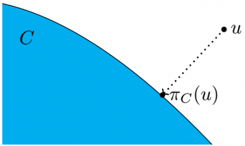
For PDE-constrained optimization under uncertainty:

- Martin, Nobile, (Krumscheid, Tsilifis) (2018, 2019, 2021)
- G. (Pflug, Scarinci, Wollner) (2019, 2020, 2021, 2023)
- Kouri, Surowiec, Staudigl (2023)
- Other recent contributions...



The projection operator

For $C \subset U$, the **projection operator** is defined by the point-to-set mapping $\pi_C\colon U \rightrightarrows C$, $u \mapsto \pi_C(u) = \operatorname{argmin}_{v \in C} \|u - v\|$.



Projection generally set-valued.

Lemma

If $C \neq \emptyset$ is closed and convex, then (1) $\pi_C\colon U \to C$ (single-valued).
(2) π_C is non-expansive, i.e., $\|\pi_C(u) - \pi_C(v)\| \leq \|u - v\|$ for all u, v.

Proof. Bauschke & Combette, Section 3.2.

Problem structures

The structure of the objective function plays a huge role in convergence proofs of iterative methods.

- **Finite sum**: $\min_{u \in U_{\mathrm{ad}}} \frac{1}{m} \sum_{i=1}^{m} J_i(u)$
  $\rightsquigarrow$ Special benefit of finite-sum structure: regular sampling of full gradient possible.
- **General (finite or infinite sum)**: $\min_{u \in U_{\mathrm{ad}}} \mathbb{E}[J(u, \xi)]$.
- **Strongly convex, convex, quasiconvex**
- **Lipschitz gradient**
- ...

(Strong) convexity for smooth objectives

Lemma

If $j\colon U \to \mathbb{R}$ is μ-strongly convex and differentiable, then

$$j(u) - j(v) \geq (\nabla j(v), u - v) + \frac{\mu}{2}\|u - v\|^2 \quad \forall u, v \in U. \tag{5}$$

In the convex and differentiable case, (5) *holds with $\mu = 0$.*

Proof. **Convex case**: we have

$$\begin{aligned} & j(\lambda u + (1-\lambda)v) \leq \lambda j(u) + (1-\lambda)j(v) \\ \Leftrightarrow \quad & \frac{j(v + \lambda(u - v)) - j(v)}{\lambda} \leq j(u) - j(v). \end{aligned}$$

Taking limit as $\lambda \to 0$, we obtain $(\nabla j(v), u - v) \leq j(u) - j(v)$.

Strongly convex case: first prove that j is strongly convex iff there exists a convex h such that $j(u) = h(u) + \frac{\mu}{2}\|u\|^2$. Then, proceed as above.

Uniqueness of minimizers in strongly convex case

Lemma

If j is μ-strongly convex and $U_{\text{ad}} \neq \emptyset$ is closed and convex, then

$$\min_{u \in U_{\text{ad}}} j(u)$$

has a unique solution.

Proof. **(for differentiable j):** optimality of $\bar{u} \in U_{\text{ad}}$ gives $(\nabla j(\bar{u}), v - \bar{u}) \geq 0$ for all $v \in U_{\text{ad}}$. Then for two optima $\bar{u}_1, \bar{u}_2$, we have

$$0 = j(\bar{u}_1) - j(\bar{u}_2) \geq \frac{\mu}{2} \|\bar{u}_1 - \bar{u}_2\|^2.$$

L-smooth functions

A function $j\colon U \to \mathbb{R}$ is called ***L*-smooth** ($L > 0$) if it is differentiable and the gradient $\nabla j\colon U \to U$ is L-Lipschitz, i.e.,

$$\|\nabla j(u) - \nabla j(v)\| \leq L\|u - v\| \quad \forall u, v \in U.$$

The set of L-smooth functions is denoted by $\boldsymbol{C_L^{1,1}(U)}$.

Lemma

If $j \in C_L^{1,1}(U)$, then

$$j(u) \leq j(v) + (j(v), u - v) + \frac{L}{2}\|u - v\|^2 \quad \forall u, v \in U. \tag{6}$$

Proof. By the fundamental theorem of calculus,

$$\begin{aligned} j(u) &= j(v) + \int_0^1 (\nabla j(v + t(u - v)), u - v)\, \mathrm{d}t \\ &= j(v) + (\nabla j(v), u - v) + \int_0^1 (\nabla j(v + t(u - v)) - \nabla j(v), u - v)\, \mathrm{d}t. \end{aligned}$$

Use Cauchy–Schwarz combined with Lipschitz condition and integrate.

L-smoothness and convexity

Lemma

If $j\colon U \to \mathbb{R}$ is convex and L-smooth, then it is also ***cocoercive****, i.e.,*

$$\frac{1}{L}\|\nabla j(u) - \nabla j(v)\|^2 \leq (\nabla j(v) - \nabla j(u), v - u). \tag{7}$$

Proof. Using convexity and L-smoothness,

$$j(u) - j(v) \pm j(z) \leq (\nabla j(u), u - z) + (\nabla j(v), z - v) + \frac{L}{2}\|z - v\|^2. \tag{$\star$}$$

Minimizing right-hand side w.r.t. z, we get $z = v - \frac{1}{L}(\nabla j(v) - \nabla j(u))$. Substituting z into ($\star$), we can verify

$$j(u) - j(v) \leq (\nabla j(u), u - v) - \frac{1}{2L}\|\nabla j(v) - \nabla j(u)\|^2. \tag{$\star\star$}$$

Claim follows by applying ($\star\star$) twice (exchanging roles of u and v).

Projected stochastic gradient (PSG) method

PSG method for solving $\min_{u\in U_{\text{ad}}}\{j(u) := \mathbb{E}[J(u,\xi)]\}$:

$$u^{n+1} = \pi_{U_{\text{ad}}}\Big(u^n - t_n G(u^n,\xi^n)\Big), \quad G(u^n,\xi^n) \approx \nabla j(u^n)$$

Notes:

- Robbins–Monro step-sizes:

$$t_n \geq 0, \quad \sum_{n=1}^{\infty} t_n = \infty, \quad \sum_{n=1}^{\infty} t_n^2 < \infty.$$

$\rightsquigarrow j(u^{n+1}) > j(u^n)$ possible: technically not a descent method!

- ξ^n is randomly chosen and independent of previous realization $\xi^1,\dots,\xi^{n-1}$.

With bias ("quasigradient"):

$$G(u^n,\xi^n) = \nabla j(u^n) + \underbrace{r^n}_{\text{bias}} + \underbrace{w^n}_{\text{mean } 0}$$

(∞-dimensional case: discretization needed).

Example

- Single realization: $\nabla_u J(u^n,\xi^n)$
- Minibatch: $\frac{1}{m_n}\sum_{i=1}^{m_n}\nabla_u J(u^n,\xi^{n,i})$
- Minibatch/single with additive bias: $\frac{1}{m_n}\sum_{i=1}^{m_n}\nabla_u J(u^n,\xi^{n,i}) + r^n$

Types of probabilistic convergence

Feature: the i.i.d. sequence $\xi^n \sim \mathbb{P}$, $n \in \mathbb{N}$, induces a discrete stochastic process $\{u^n\} \rightarrow$ probabilistic convergence statements.

$$\xrightarrow{L^s} \;\underset{s>r\geq 1}{\Rightarrow}\; \xrightarrow{L^r}$$

$$\qquad\qquad\Downarrow$$

$$\xrightarrow{\text{a.s.}} \;\Rightarrow\; \xrightarrow{p} \;\Rightarrow\; \xrightarrow{d}$$

Figure: Relationship between convergence in L^p, almost sure (a.s.), probability (p), and distribution (d).

Typical for SA: proofs of L^2 **convergence** and **almost sure convergence**.

- **Convex case**: convergence of $\{u^n\}$ to set of minimizers, convergence of $\{j(u^n)\}$.
- **Nonconvex case**: convergence of stationarity measure, convergence of $\{j(u^n)\}$.

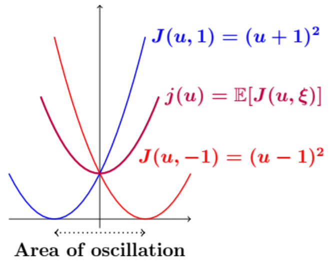
Nonconvergence with constant step-sizes

Example

Let $J(u,\xi) := (u+\xi)^2 \Rightarrow \nabla_u J(u,\xi) = 2(u+\xi)$ and $\xi = \pm 1$ (with equal probability). The minimizer of

$$\mathbb{E}[J(u,\xi)] = \frac{1}{2}((u+1)^2 + (u-1)^2)$$

is $\bar{u} = 0$.



Suppose $t_n = \alpha > 0$ is constant.

We prove: $|u^n| < \varepsilon \Rightarrow |u^{n+1}| > \varepsilon$ for $\varepsilon < \frac{\alpha}{1-\alpha}$.

Case $\xi = -1$:
$$u^{n+1} = u^n - \alpha \nabla_u J(u^n, -1) \geq -\varepsilon(1-2\alpha) + 2\alpha > \varepsilon.$$

Case $\xi = 1$: Analogous argument. Thus $u^n \not\to 0$.

- Example explains why Armijo backtracking fails; variance from stochastic gradient needs to be damped! Two basic strategies: increase batch sizes and/or use decreasing step-sizes.

Python script: `https://nbviewer.org/url/caroline.geiersbach.com/PSGD.ipynb`

Assumptions for convergence

1. **Constraint set**: $U_{\text{ad}} \neq \emptyset$ is closed and convex.
2. **Convexity, smoothness, boundedness of expectation**.
3. **Measurability**: $\{u^n\}$ and $\{r^n\}$ are $\mathcal{F}_n$-measurable. [22]
4. **Bias decays fast enough:** $\sum_{n=1}^{\infty} t_n \|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)} < \infty$ and $\sup_n \|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)} < \infty$ are satisfied.
5. **Growth condition**: There exist $M_1, M_2 \geq 0$ such that $\mathbb{E}[\|G(u,\xi)\|^2] \leq M_1 + M_2\|u\|^2$ for all $u \in U_{\text{ad}}$.

[22] Measurability can be argued if G and j are continuous and U is separable.

Almost sure convergence for convex problems

Theorem (Convergence for convex j)

Under assumptions,

1. $\{j(u^n)\}$ ***converges a.s. to*** *the minimum,*
2. $\{u^n\}$ ***weakly converges a.s. to some minimizer****,*
3. *Strongly convex case:* $\{u^n\}$ ***converges strongly a.s.*** *to the unique minimizer.*

u^n converges "weakly a.s." to $\bar{u}$ if $\mathbb{P}(\{\omega \in \Omega \mid \lim_{n\to\infty} \langle v, u^n \rangle = \langle v, \bar{u} \rangle \quad \forall v \in U\}) = 1$.

Notes about proof technique

The **natural filtration** is denoted by $\mathcal{F}_n = \sigma(\{\xi^1, \dots, \xi^{n-1}\})$, $n \in \mathbb{N}$. [23] The *conditional expectation* $\mathbb{E}[u|\mathcal{F}_n]$ of a random vector u is a random vector such that

$$\int_A \mathbb{E}[u|\mathcal{F}_n](\omega) \, \mathrm{d}\mathbb{P}(\omega) = \int_A u(\omega) \, \mathrm{d}\mathbb{P}(\omega) \quad \forall A \in \mathcal{F}_n.$$

Thus, if u is $\mathcal{F}_n$-measurable, we have $\mathbb{E}[u|\mathcal{F}_n] = u$ a.s. Also, since $\Omega \in \mathcal{F}_n$, $\mathbb{E}[\mathbb{E}[u|\mathcal{F}_n]] = \mathbb{E}[u]$ (law of total expectation).

Lemma (Robbins–Siegmund [24])

Suppose $\{\mathcal{F}_n\}$ is an increasing sequence of σ-algebras and v_n, a_n, b_n, c_n are $\mathcal{F}_n$-measurable nonnegative random variables. If

$$\mathbb{E}[v_{n+1}|\mathcal{F}_n] \le v_n(1 + a_n) + b_n - c_n, \qquad \sum_{n=1}^\infty a_n < \infty, \qquad \sum_{n=1}^\infty b_n < \infty,$$

then $\{v_n\}$ converges and $\sum_{n=1}^\infty c_n < \infty$ a.s.

---

[23] $\sigma(\{\xi^1, \dots, \xi^n\})$ is the smallest σ-algebra such that ξ^i is measurable for all i.

[24] Robbins, Siegmund. **A convergence theorem for non negative almost supermartingales and some applications** (1971).

Efficiency estimates - strongly convex case

Rules for bias and step-sizes for μ-strongly convex objective:

$$\|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)} \leq \frac{\kappa}{n+\nu}, \quad t_n = \frac{\theta}{n+\nu},$$

where $\theta > 1/(2\mu)$, and $\nu \geq 2\theta\kappa/(2\mu\theta - 1) - 1$, and $\kappa > 0$.

Efficiency estimates:

$$\mathbb{E}[\|u^n - \bar{u}\|] \leq \sqrt{\frac{\rho}{n+\nu}}, \qquad \text{where } \rho = \rho(\nu, \mu, \theta, \|u^1 - \bar{u}\|, \kappa, M_1, M_2).$$

If additionally $j \in C^{1,1}_L$ and $\nabla j(\bar{u}) = 0$, then

$$\mathbb{E}[j(u^n) - j(\bar{u})] \leq \frac{L\rho}{2(n+\nu)}.$$

$\rightsquigarrow$ **Same order of convergence as for bounded U_{ad}, unbiased case!**

Poor convergence in convex case

Example ([25])

- **Poor choice of step size**: Consider $j(u) = \frac{1}{10}u^2$, $U_{\text{ad}} = [-1,1]$, $G(u,\xi) = \nabla j(u)$. Suppose that we choose $\theta = 1$ (here we have $\mu = \frac{1}{5}$). Then,

$$u^n = \prod_{s=1}^{n-1}\left(1 - \frac{1}{5s}\right) = \exp\left\{-\sum_{s=1}^{n-1}\ln\left(1+\frac{1}{5s-1}\right)\right\} > \exp\left\{-\sum_{s=1}^{n-1}\frac{1}{5s-1}\right\}$$
$$> \exp\left\{-\left(0.25 + \int_1^{n-1}\frac{1}{5t-1}\,\mathrm{d}t\right)\right\} > 0.8n^{-1/5}.$$

At iterate $n = 10^9$, the iterated solution is greater than 0.015 (the solution is $\bar{u} = 0$).

- **Convex (not strongly convex) case**: Consider $j(u) = u^4$, $U_{\text{ad}} = [-1,1]$, $G(u,\xi) = \nabla j(u)$, $t_n = \frac{\theta}{n}$, $0 < u_1 \leq \frac{1}{6\sqrt{\theta}}$. Then,

$$u^n \geq \frac{u^1}{\sqrt{1 + 32\theta(u^1)^2(1+\ln(n+1))}}.$$

Remedy: **iterate averaging** [26]

[25] Nemirovski, Juditsky, Lan, Shapiro. **Robust stochastic approximation approach to stochastic programming** (2009).

[26] Polyak and Juditsky. **Acceleration of stochastic approximation by averaging** (1992).

Efficiency estimates with iterate averaging

Iterate averaging: run PSG with larger steps; compute the **running average** of the iterates:

$$\tilde{u}_i^N = \sum_{n=i}^{N} \gamma_n u^n, \qquad \gamma_n := t_n / (\sum_{\ell=i}^{N} t_\ell).$$

Rules for bias and step-sizes (U_{ad} bounded):

$$\sum_{n=i}^{N} \frac{\|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)}}{\sqrt{n}} \propto 1, \quad t_n = \frac{\theta D_{\text{ad}}}{\sqrt{Mn}}, \qquad \theta > 0,\, D_{\text{ad}} := \max_{u \in U_{\text{ad}}} \|u^1 - u\|, \text{ and } M = M_1.$$

Efficiency estimate for $i = \lceil \alpha N \rceil$ for some $\alpha \in (0,1)$:

$$\mathbb{E}[j(\tilde{u}_i^N) - j(u)] \le \frac{\rho}{\sqrt{N}} \qquad \rho = \rho(\theta, M, D_{\text{ad}}, \|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)}).$$

Note: for unbounded U_{ad}, possible to show that convergence is nearly $\mathcal{O}(1/\sqrt{N})$.

Finite element method (FEM) in 2D

Weak formulation of model problem:

Find $y = y(\xi) \in V := H_0^1(D)$ such that

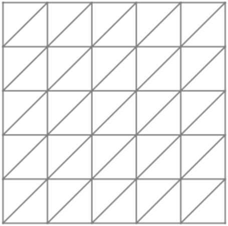
$$\int_D a(x,\xi)\nabla y(x) \cdot \nabla v(x)\,\mathrm{d}x = \int_D u(x)v(x)\,\mathrm{d}x \quad \forall v \in V.$$

Finite element approximation:

- Triangulate D into elements $\mathcal{T}_h = \{T_i\}$.
- Choose basis $\{\psi_j\} \subset V_h \subset V$.
- Stiffness matrix:

$$A_{k\ell} = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j} a_{ij}\partial_i\psi_k\partial_j\psi_\ell\,\mathrm{d}x.$$

- Load vector: $l_i = \int_D u\psi_i\,\mathrm{d}x$.
- Solve: $AY = l$.



Triangulation $\mathcal{T}_h$ showing triangles $(T_i)_{i=1,\dots,50}$ defined by edges and nodes.

Discretization

State/adjoint equations - piecewise linear finite elements:

$$Y_h := \{v \in H^1(D) \mid v|_T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}_h\},$$
$$Y_h^0 := Y_h \cap H_0^1(D).$$

Controls - discretization using piecewise constant finite elements:

$$U_h := \{u \in L^2(D) \mid v|_T \in \mathcal{P}_0(T) \text{ for all } T \in \mathcal{T}_h\}, \quad U_{\text{ad},h} = U_h \cap U_{\text{ad}}.$$

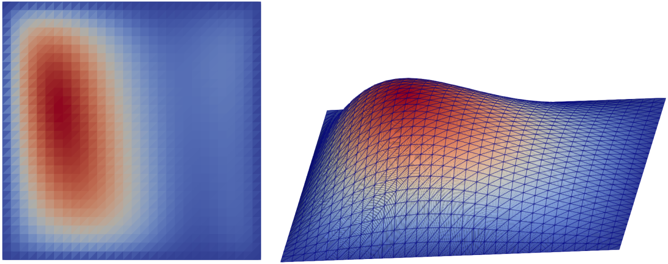


Figure: Example $u_h \in U_{\text{ad},h}$ (left) and $y_h \in Y_h^0$ (right).

Discretization of problem and stochastic gradient

The (spatially) discretized version of the model problem:

$$\begin{aligned} \min_{u \in U_{\mathrm{ad},h}} \ & \frac{1}{2}\mathbb{E}\left[\|y_h - y_d\|^2_{L^2(D)}\right] + \frac{\mu}{2}\|u_h\|^2_{L^2(D)} \\ \text{s.t.} \quad & \int_D I_h a(\xi) \nabla y \cdot \nabla v \, \mathrm{d}x = \langle u_h, v_h \rangle_{L^2(D)} \quad \forall v_h \in Y_h^0. \end{aligned} \tag{P$'_h$}$$

$I_h \ldots$ interpolation into element wise constant or linear finite elements.

Stochastic gradient with bias (numerical error):

$$\boldsymbol{\nabla_u J_h(u_h, \xi) = \mu u_h + P_h p_h(\xi),}$$

where $p_h(\xi) \in Y_h^0$ solves the PDE

$$\int_D I_h a(\xi) \nabla p_h(\xi) \cdot \nabla v_h \, \mathrm{d}x = (y_h(\xi) - y_d, v_h)_{L^2(D)} \quad \forall v_h \in Y_h^0.$$

$P_h \colon U \to U_h \ldots$ L^2-projection, defined for $v \in L^2(D)$ by $P_h(v)\big|_T = \frac{1}{|T|}\int_T v \, \mathrm{d}x$.

Error in stochastic gradient

Error in the stochastic gradient can be split as follows:

$$\nabla j(u_h) = \underbrace{\mu u_h^n + P_h p_h^n(\xi^n)}_{=:\nabla_u J_h(u_h^n,\xi^n)} + \underbrace{\mathbb{E}[p^n(\xi)] - p^n(\xi^n)}_{=:w^n} + \underbrace{p^n(\xi^n) - P_h p_h^n(\xi^n)}_{=:r^n}$$

Sketch of estimate for $K_n = \|p^n(\xi) - P_h p_h^n(\xi)\|_{L_P^\infty(\Xi,U)}$, $U = L^2(D)$:

$$\begin{aligned}
\|p^n(\xi) - P_h p_h^n(\xi)\|_U &\le \|P_h p_h^n(\xi) - P_h p^n(\xi)\|_U + \|p^n(\xi) - P_h p^n(\xi)\|_U \\
&\le \|p_h^n(\xi) - p^n(\xi)\|_U + ch\|\nabla p^n(\xi)\|_U \quad \textbf{(Projection + Error for } P_h\textbf{)} \\
&\le \|p_h^n(\xi) - p^n(\xi)\|_U + ch\Big(\|y_d\|_U + \|u_h\|_U\Big) \quad \textbf{(Stability of } p(\xi) \textbf{ and } y(\xi)\textbf{)} \\
&\le ch^{\min(2s,t)}\Big(\|y_d\|_U + \|u_h\|_U\Big) + ch\Big(\|y_d\|_U + \|u_h\|_U\Big) \quad \textbf{(Aubin-Nitsche trick [27])}
\end{aligned}$$

$\Rightarrow$ **Bound for bias:** $K_n \le ch^{\min(2s,t,1)}$.

[27] For $a(\xi) \in C^t(\bar{D})$, $\exists s_0 \in (0,t]$: for any $0 \le s < s_0$, any $u \in H^{s_0-1}(D)$, it holds that $\|y(\xi)\|_{H^{1+s}(D)} \le C_s\|u\|_{H^{s-1}(D)}$.

Mesh refinement rule

For the **strongly convex case**, we get with the requirement that $K_n \leq \frac{K}{n+\nu}$ the rule

$$t_n = \frac{\theta}{n+\nu}, \quad h_n \leq \left(\frac{c}{n+\nu}\right)^{1/\min(2s,t,1)}. \tag{8}$$

For the **convex case**, we get with the requirement $\sum_{n=i}^N \frac{K_n}{\sqrt{n}} \propto 1$ the rule

$$t_n = \frac{\theta D_{\mathrm{ad}}}{\sqrt{Mn}}, \quad h_n \leq \left(\frac{c}{\sqrt{n}+\sqrt{n-1}}\right)^{1/\min(2s,t,1)}. \tag{9}$$

PSG with mesh refinement for (P'_h)

Initialization: Select $h_1 > 0$, $u_h^1 \in U_{\mathrm{ad},h}$
for $n = 1, 2, \ldots$ **do**
  if $h = h_n$ **is too large per** (8) **or** (9) **then**
    Refine mesh $\mathcal{T}_{h_n}$ **until** $h = h_n$ **is small enough.**
  end if
  Given new sample ξ^n, calculate (y_h^n, p_h^n) by solving corresponding state, adjoint equations.
  $u_h^{n+1} := \pi_{U_{\mathrm{ad},h}}(u_h^n - t_n(\lambda u_h^n + P_h p_h^n))$.
end for

Obtained solutions

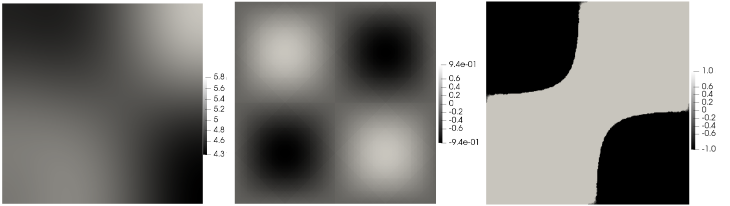


Figure: "Smooth field" (left) with control in strongly convex setting (middle) and convex setting (right).

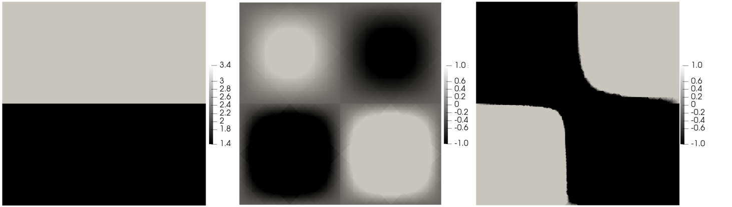


Figure: "Piecewise constant field" (left) with control in strongly convex setting (middle) and convex setting (right).

Convergence behavior: strongly convex case

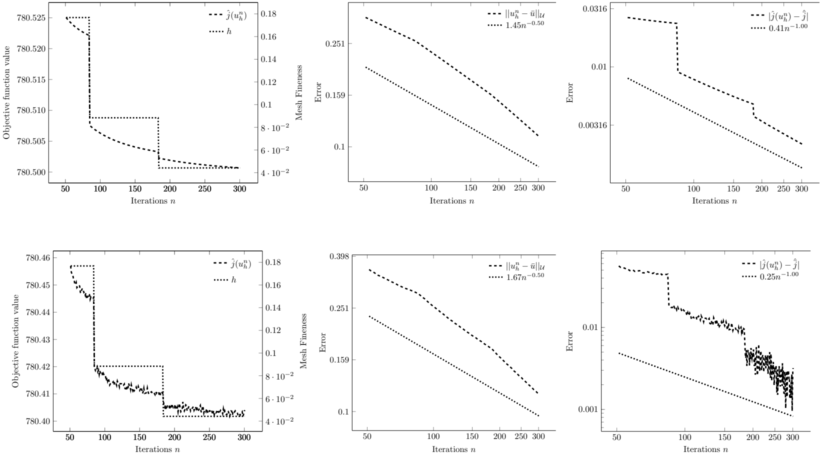


Figure: Smooth random field (top row) and piecewise constant random field (bottom row).

Convergence behavior: convex case

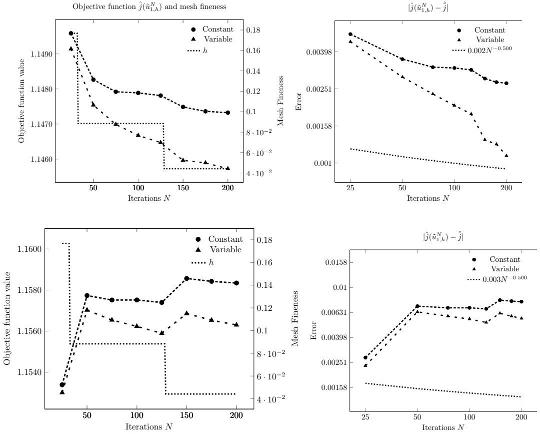


Figure: Smooth random field (top row) and piecewise constant random field (bottom row).

→ Reduced convergence rate for piecewise constant a!

Convergence behavior: convex case

.... Expected convergence rate can be achieved by choosing a more aggressive mesh refinement strategy. Here, we chose $\min(2s, t, 1) = 0.5$**.**

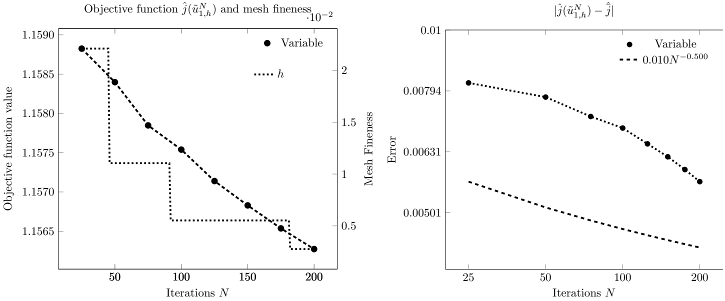
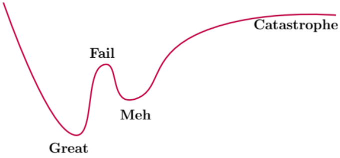


Figure: Piecewise constant random field using variable step-size rule.

Nonconvex objective

Motivation: in many applications, j is nonconvex (e.g., if $u \mapsto S(u,\omega)$ is nonlinear). What can be said about convergence of SA for nonconvex problems?



Without convexity, one can only expect to converge to stationary points.
$\rightsquigarrow$ Need to ensure iterates $\{u^n\}$ are bounded along with some smoothness assumption on ∇j.

Minimal assumption: objective has L-Lipschitz gradients, so that we have inequality

$$j(u) \leq j(v) + \langle \nabla j(v), u - v \rangle + \frac{L}{2}\|u - v\|^2 \quad \forall u, v \in U. \tag{10}$$

Assumptions for convergence

SG method for solving unconstrained $\min_{u\in U}\{j(u) := \mathbb{E}[J(u,\xi)]\}$:

$$u^{n+1} = u^n - t_n G(u^n, \xi^n), \qquad \text{with RM rule:} \quad t_n \geq 0, \quad \sum_{n=1}^{\infty} t_n = \infty, \quad \sum_{n=1}^{\infty} t_n^2 < \infty.$$

Assumptions - SG with bias: $G(u^n, \xi^n) = \nabla j(u^n) + r^n + w^n$

- **Iterates are bounded** [28]**.**
- $j \in C_L^{1,1}$ **is bounded below.**
- **Measurability**: $\{u^n\}$ and $\{r^n\}$ are $\mathcal{F}_n$-measurable.
- **Bias decays fast enough:** $r^n = \mathbb{E}[G(u^n,\xi^n)|\mathcal{F}_n] - \nabla j(u^n)$ is $\mathcal{F}_n$-measurable for all n and $\sum_{n=1}^{\infty} t_n \|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)} < \infty$ and $\sup_n \|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)} < \infty$ are satisfied.
- **Growth condition:** There exists a function $M\colon U \to [0,\infty)$, that is bounded on bounded sets, such that $\mathbb{E}[\|G(u,\xi)\|^2] \leq M(u)$.

[28] Boundedness of iterates can be shown under additional growth conditions on the gradient.

Almost sure convergence result

Theorem (Convergence for $j \in C_L^{1,1}$)

1. *The sequence* $\{j(u^n)\}$ ***converges a.s.*** & $\liminf_{n\to\infty} \|\nabla j(u^n)\| = 0$ ***a.s.***
2. *If* $F(u) := \|\nabla j(u)\|^2$ *satisfies* $F \in C_{L_F}^{1,1}(U)$*, then* $\lim_{n\to\infty} \nabla j(u^n) = 0$ ***a.s.***

Note: No convergence statement for iterates.

Proof of first claim

Since $j \in C_L^{1,1}$ and $g^n = \nabla j(u^n) + r^n + w^n$,

$$j(u^{n+1}) \overset{(10)}{\leq} j(u^n) - t_n(\nabla j(u^n), g^n) + \frac{Lt_n^2}{2}\|g^n\|^2$$

$$\overset{\text{Monotonicity of } \mathbb{E}[\cdot|\mathcal{F}_n]}{\Longrightarrow} \quad \mathbb{E}[j(u^{n+1})|\mathcal{F}_n] \leq j(u^n) - t_n(\nabla j(u^n), \overbrace{\mathbb{E}[g^n|\mathcal{F}_n]}^{=\nabla j(u^n)+r_n}) + \frac{Lt_n^2}{2}\mathbb{E}[\|g^n\|^2|\mathcal{F}_n].$$

Apply Robbins–Siegmund lemma.

Proof of second claim

Since $F \in C^{1,1}_{L_F}$,

$$|\mathbb{E}[\underbrace{F(u^{n+1})}_{=:v_{n+1}} - F(u^n)|\mathcal{F}_n]| \le \left| -t_n(\nabla F(u^n), \mathbb{E}[g^n|\mathcal{F}_n]) + \frac{L_F t_n^2}{2}\mathbb{E}[\|g_n\|^2|\mathcal{F}_n] \right|$$

$$\le |-t_n(2(\nabla^2 j(u^n))^* \nabla j(u^n), \nabla j(u^n) + r^n)| + \frac{L_F M(u^n) t_n^2}{2}$$

$$\le 2L t_n \|\nabla j(u^n)\|^2 + 2LM_1 t_n \|r^n\|_{L^\infty_{\mathbb{P}}(\Omega)} + \frac{L_F M(u^n) t_n^2}{2}.$$

Result follows with following theorem.

Lemma (Quasimartingale convergence theorem)

Suppose $\{\mathcal{F}_n\}$ is an increasing sequence of σ-algebras and v_n is a $\mathcal{F}_n$-measurable random variable. If

$$\sup_n \mathbb{E}[\max\{0, -v\}] < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \mathbb{E}\Big[|\mathbb{E}[v_{n+1} - v_n|\mathcal{F}_n]|\Big] < \infty,$$

then $\{v_n\}$ converges a.s. to a $\mathbb{P}$-integrable random variable v_∞.

Convergence rates?

In the nonconvex case, it's possible to show that if t_n is decreasing and $\sum_{j=1}^{\infty} \frac{t_j}{\sum_{k=1}^{j} t_k} = \infty$ [29] then, almost surely,

$$\min_{t=1,\dots,n} \|\nabla j(u^t)\|^2 = o\left(\frac{1}{\sum_{j=1}^{n} t_j}\right).$$

Poor scaling of the step size in the nonconvex setting $\to$ convergence may be very slow...

Figure: Heat death of the universe?

Scaling of

$$t_n = \theta / n^s$$

obtained by intuition and offline tuning.

[29] Satisfied for e.g. $t_n = \theta / n^s$ with $\theta > 0$ and $s \in (0.5, 1]$.

Model problem

Optimal control of semilinear elliptic equation under uncertainty

$$\min_{u \in L^2(D)} \quad \frac{1}{2}\mathbb{E}[\|y(\xi) - y_d\|^2_{L^2(D)}] + \frac{\mu}{2}\|u\|^2_{L^2(D)}$$

$$\text{s.t.} \quad -\nabla \cdot (a(x,\xi)\nabla y(x)) + y(x) + y^5(x) = u(x), \qquad x \in D \quad \text{a.s.,}$$

$$\frac{\partial y}{\partial n} = 0, \qquad x \in \partial D \quad \text{a.s.}$$

- **Nonlinear** control-to-state map $u \mapsto S(u,\omega)$.
- Good performance with step size $t_n = \theta/n$ with $\theta = 2/\mu$ informed by strongly convex objective.

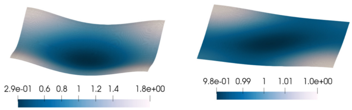


Figure: Computed optimal control (left) and state (right) for $\mu = 1$.

Convergence behavior

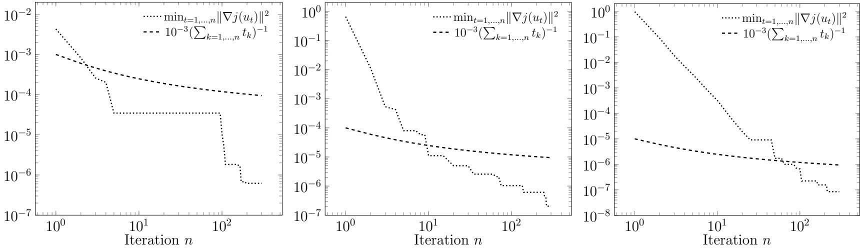


Figure: Convergence for $\mu = 1$ (left), $\mu = 0.1$ (middle), $\mu = 0.01$ (right).

Performance outperforms theory $\rightsquigarrow$ nonconvexity is rather harmless for this problem.

Evidence of mesh independence

Idea: run algorithm on different meshes and compare # of iterations needed until the (estimated!) residual $\hat{r}_N$ reaches a tolerance `tol`.

Hope: # of iterations stays stable → *independent* of the discretization.

h	# triangles	objective function $\hat{f}_N$	# iterations N until $\hat{r}_N \leq$ `tol`
$7.1e^{-2}$	800	$4.160e^{-2}$	191
$4.7e^{-2}$	1800	$4.157e^{-2}$	295
$3.5e^{-2}$	3200	$4.157e^{-2}$	233
$2.8e^{-2}$	5000	$4.156e^{-2}$	257
$2.4e^{-2}$	7200	$4.156e^{-2}$	271
$2.0e^{-2}$	9800	$4.155e^{-2}$	251

Mesh independence test for SA + FEM method showing evidence of mesh independence.

Example for numerics - with state constraints

Modification of problem (P) to include state constraint "$y(\cdot,\xi) \le \alpha$": Moreau–Yosida regularized problem is

$$\begin{aligned}\min_{u\in U_{\text{ad}}} \quad & \frac{1}{2}\mathbb{E}[\|y-y_d\|^2_{L^2(D)}] + \frac{\mu}{2}\|u\|^2_{L^2(D)} + \frac{\gamma}{2}\mathbb{E}[\|\max(0,y-\boldsymbol{\alpha})\|^2_{L^2(D)}] \\ \text{s.t.} \quad & -\nabla\cdot(a(x,\xi)\nabla y(x,\xi)) = u(x), \quad \text{on } D\times\Xi, \\ & y(x,\xi) = 0, \quad \text{on } \partial D\times\Xi.\end{aligned}$$

With $J(u,\xi) = \frac{1}{2}\|y-y_d\|^2_{L^2(D)} + \frac{\mu}{2}\|u\|^2_{L^2(D)} + \frac{\gamma}{2}\|\max(0,y-\boldsymbol{\alpha})\|^2_{L^2(D)}$, stochastic gradient can be computed by:

$$\begin{aligned}D_u J(u,\xi)[h] &= \langle y-y_d, A^{-1}(\xi)h\rangle + \langle \mu u, h\rangle + \langle \gamma\max(0,y-\boldsymbol{\alpha}), A^{-1}(\xi)h\rangle \\ &= \langle \underbrace{\underbrace{A^{-*}(\xi)(y-y_d)+\gamma\max(0,y-\boldsymbol{\alpha}))}_{=:p(\xi)} + \mu u}_{=:G^{\gamma}(u,\xi)}, h\rangle.\end{aligned}$$

Path-following PSG method

Initialization: Select γ_1, $h_1 > 0$, $u_h^1 \in U_{\text{ad},h}$.
for $m = 1, 2, \ldots$ **do**
  $u_h^{m_n} \leftarrow$ Run PSG with mesh refinement with gradient $G_h^{\gamma_m}(u_h^n, \xi^n)$ for m_n steps.
  Increase γ_m, choose $h = h_1$, project u_h^{n+1} onto mesh h_1.
end for

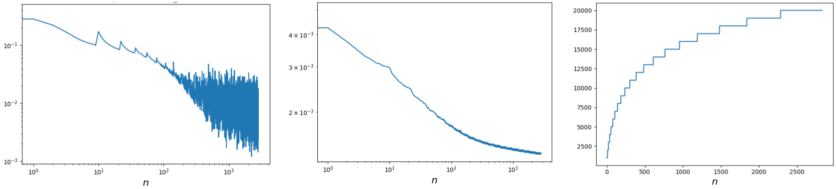


Figure: Plot of stationarity measure $\|G^{\gamma_n}(u^n, \xi^n)\|_{L^2}^2$ (left); infeasibility measure $\|\max(0, y^n - \alpha)\|_{L^2}^2$ (middle); sequence of γ_n (right).

Notes:

- U_{ad} chosen large enough so that $\|G^{\gamma_n}(u^n, \xi^n)\|_{L^2(D)}^2$ is a stationarity measure.
- Variance reduction (increased batch sizes) may improve performance for larger γ_n.

Main takeaways

- Introduction to stochastic optimization on Banach spaces.
- Optimality conditions in reducible form and challenges in irreducible form.
- Case study from PDE-constrained optimization under uncertainty.
- Results in Hilbert-valued stochastic approximation
  - It is possible to show almost sure convergence in Hilbert spaces even with (decreasing) numerical error, unbounded U_{ad}, and/or nonconvex functions.
- SA applicable to more involved problems (nonconvexity, nonsmoothness) but further development is needed!

  $\rightsquigarrow$ Many open questions for further research: optimality conditions, models with risk measures, higher-order and accelerated methods, line search methods for nonconvex problems, more robust methods for state constraints, convergence of SA for nonconvex problems in Banach spaces . . .

References

For technical details behind theory/numerics in this lecture, see:

- SA - convex setting
  - **G.**, Pflug. **PSGM for convex constrained probs. in Hilbert spaces.** SIOPT (2019).
  - **G.**, Wollner. **SGM with mesh refinement.** SISC (2020).
- SA - nonconvex setting
  - **G.**, Scarinci. **SPGM for nonconvex probs. in Hilbert space.** COAP (2021).
  - **G.**, Scarinci. **SGM for nonconvex probs. in Hilbert space.** JDA (2023).
- Optimality conditions for probabilistic state constraints
  - **G.**, Wollner. **Optimality conditions for almost sure state constraints.** SIOPT (2021).
  - **G.**, Hintermüller. **Moreau–Yosida regularization for almost sure state constraints.** ESAIM: COCV (2022).
  - **G.**, Henrion. **Optimality conditions with chance constraints.** Math. Oper. Res. (2024).

Other recommended reading:

- Heinkenschloss and Kouri. **Optimization problems governed by systems of PDEs with uncertainties** (2025).
- Kouri and Surowiec. **Existence and optimality conditions for risk-averse PDE-constrained optimization** (2018).
- Shapiro, Dentcheva, and Ruszczyński. **Lectures on stochastic programming: modeling and theory.** (2009).
- Bottou, Curtis, and Nocedal. **Optimization methods for large-scale machine learning** (2018).

Cartoon images in this lecture were generated using ChatGPT.


THANK YOU
FOR YOUR
ATTENTION!